EMPOWERING MATHEMATICS LEARNERS

Yearbook 2017
Association of Mathematics Educators
EMPOWERING MATHEMATICS LEARNERS

Yearbook 2017
Association of Mathematics Educators

editors
Berinderjeet Kaur
LEE Ngan Hoe
Nanyang Technological University, Singapore

World Scientific
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Chapter 1

Empowering Mathematics Learners

Berinderjeet KAUR                         LEE Ngan Hoe

This chapter provides a context for empowerment in mathematics teaching and learning in Singapore schools. It also provides an overview of the chapters in the book. The chapters are grouped into four sections which are about empowering mathematics learners. The sections focus on how learners could be empowered in their learning i) through mathematical content, ii) through cognitive and affective processes, iii) through purposefully designed mathematical tasks, and iv) whilst developing 21st century competencies. These aspects of empowerment also reflect the various important aspects of the teaching and learning processes – the content, the learner, the teacher, and the context.

1 Introduction

This yearbook of the Association of Mathematics Educators (AME) in Singapore focuses on Empowering Mathematics Learners. Like past yearbooks, Mathematical Problem Solving (Kaur, Yap, & Kapur, 2009), Mathematical Applications and Modelling (Kaur & Dindyal, 2010), Assessment in the Mathematics Classroom (Kaur & Wong, 2011), Reasoning, Communication and Connections in Mathematics (Kaur & Toh, 2012), Nurturing Reflective Leaners in Mathematics (Kaur, 2013), Learning Experiences to Promote Mathematics Learning (Toh, Toh, & Kaur, 2014), and Developing 21st Century Competencies in the Mathematics Classroom (Toh & Kaur, 2016) the theme of this book is shaped by the school mathematics curriculum developed by the Ministry
Empowering Mathematics Learners

of Education (MOE) and the needs of mathematics teachers in Singapore schools.

The theme of this yearbook is related to the idea of ‘empowerment’. According to the internet dictionary, dictionary.com, the two distinct meanings of the verb, ‘empower’ are:

1. to give power or authority to; authorize, especially by legal or official means:
   *I empowered my agent to make the deal for me.*
   *The local ordinance empowers the board of health to close unsanitary restaurants.*

2. to enable or permit:
   *Wealth empowered him to live a comfortable life.*
   *(www.dictionary.com)*

Both of these meanings are applicable in the context of the school mathematics curriculum and mathematics teaching pedagogy in Singapore. Several initiatives of the Ministry of Education (MOE) in Singapore, for example the framework for 21st century competencies and student outcomes (MOE, 2010) and the decision some years ago by the curriculum authorities in Singapore to sanction the use of technology especially calculators for teaching and learning mathematics and for examinations illustrate the first meaning of “empower”. Referring to the saying:

   *Give a man a fish and you feed him for a day;*
   *Teach a man to fish and you feed him for a lifetime.*
   *(Anon).*

when teachers teach their students “to fish” whilst enacting the school mathematics curriculum they are illuminating the second meaning of “empower”.

To reinforce awareness amongst mathematics teachers in Singapore about empowerment and mathematics learning, the Association of Mathematics Educators and the Mathematics and Mathematics Education
Academic Group at the National Institute of Education chose the theme: Empowering Mathematics Learners for their 2016 annual conference. The following 17 peer-reviewed chapters in this book are a result of keynote lectures and workshops that were part of the scientific programme of the conference.

The chapters in the book are grouped into four sections. All the sections are about empowering mathematics learners. The sections focus on how learners could be empowered in their learning i) through mathematical content, ii) through cognitive and affective processes, iii) through purposefully designed mathematical tasks, and iv) whilst developing 21st century competencies. The aspects of empowerment, though listed in four different sections, are inter-related. Consequently chapters in each section may also contain issues that are addressed in other sections; the section that each chapter is placed under only reflects its primary emphasis. These aspects of empowerment also reflect the various important aspects of the teaching and learning processes – the content, the learner, the teacher, and the context.

2 Empowering Learners through Mathematical Content

In Singapore schools, teachers are key to student learning. They take on a myriad of roles to facilitate the learning of mathematics by their charges. There are four chapters in this section. Three of the chapters illustrate how mathematical content may be developed during instruction such that learners are empowered in their acquisition of knowledge, while one chapter examines preparation of mathematics teachers so that their mathematical knowledge is robust to empower mathematics learners. Chapter 2, by Chua, illustrates a teaching unit purposefully designed to help teachers develop the topics of expansion and factorisation in a more student-centred way, and with greater reliance on student comprehension and explanation. Not only do the activities in the teaching unit hope to support students in developing the expanding and factorising skills and in appreciating the inverse relationship between expansion and factorisation, but also to encourage them to assume greater responsibility in their own learning and to become self-directed and independent learners.
Tan and Hang, in chapter 3, draw on a learning study with the goal of developing students’ mathematical reasoning competencies for the topic conics and illustrate how classroom practices of mathematics teachers can contribute to students’ mathematical noticing through cognitive offloading, re-orienting and instantiation. Kissane in chapter 4, describes how graphic calculators can empower students in their learning of A-Level mathematics, for topics such as matrices, functions and graphs through representations, computations, explorations, and affirmation of their thinking. Teachers of mathematics must possess sound mathematical knowledge themselves to engage their learners in learning mathematics. Cho and Kwon, in chapter 5, advocate that for teachers to teach school mathematics conceptually and meaningfully, they must develop a rigorous understanding of mathematical theorems, an understanding of extension of mathematical definitions, and a rigorous understanding of mathematical definitions.

3 Empowering Learners through Cognitive and Affective Processes

Teachers engage students in both cognitive and affective processes during classroom instruction to facilitate their learning of mathematics. The four chapters in this section show how teachers may use specific strategies to do so. Anthony and Hunter, in chapter 6, demonstrate how three instructional practices enable teachers to develop student voice in mathematics classrooms to empower the students through cognitive and affective processes. The practices are i) student engagement in rich mathematical discourse through group activities, ii) teacher noticing and valuing of their students’ thinking and use of their thinking as a resource for learning, and iii) teachers positioning their students as competent. In chapter 7, Wong, proposes a 5-stage information processing model that can help students develop effective memory strategies thereby empowering them to enhance their mathematical performance. The framework covers ways of acquiring, processing, encoding, strengthening, and retrieving information encountered during mathematics learning seeking to empower the students through cognitive processes.
Empowering Mathematics Learners

Yeo KK, in chapter 8, emphasises the need for primary school students to use multiple representations to solve process problems. He illustrates through the solutions of three process problems, how teachers may use multiple representations in their lessons to empower students through cognitive processes in developing flexible connections among the multiple modes of representations. Lastly, in chapter 9, Loh and Lee illustrate how different types of student activities may lead students to exhibit a range of cognitive and metacognitive strategies in mathematical problem solving. Such insights help teachers to better structure their instructions thereby developing empowered and self-directed learners through metacognitive processes.

4 Empowering Learners through Purposeful Mathematical Tasks

Experiences gained by students through the mathematical tasks, that teachers engage them in to actualise the intended curriculum, form the bedrock of their knowledge and perception of mathematics. The five chapters in this section show i) how teachers can empower learners through their choices and design of purposeful mathematical tasks teachers, and ii) the knowledge teachers themselves need to have to enact worthy mathematical tasks. Toh, in chapter 10, demonstrates how teachers may craft purposeful tasks and enact them using a problem solving framework to empower learners in their mathematics lessons. Toh draws on typical textbook tasks and shows how teachers can craft mathematical tasks that give students more scope in decision making. In chapter 11, Wijaya, shows how exploratory tasks together with appropriate scaffolding provided by the teacher can empower learners of mathematics. Wijaya also shows how a regular textbook task can be modified into an exploratory task. Yeo BW, in chapter 12, discuss how teachers can use open and guided mathematical investigative tasks to empower secondary school students to think and solve problems like mathematicians. The examples in the chapter provide teachers with a resource to draw on for investigative tasks.

Veloo and Parmjit, in chapter 13, share with readers a number of their studies that illustrate the power of representations in children’s solutions
of problems through task design. They urge teachers to adopt a conceptual approach to the learning of mathematics by empowering their students to use representations. Ng, in chapter 14, reports on an exploratory study which investigated the use of open-ended real-world tasks in primary mathematics classrooms. The implications of the study, particularly empowering teachers of mathematics to i) understand the role of assumption making in real-world problems, ii) differentiate between assumptions and conditions, and iii) work with open-ended problems with perceived “missing information”, are valuable insights for teacher development in the design of such tasks.

5 Empowering Learners through 21st Century Competencies

The framework for 21st century competencies and student outcomes for Singapore schools (MOE, 2010) aims to develop a young person into a confident person, a self-directed learner, an active contributor and a concerned citizen. In this section the four chapters show how students may be empowered in their learning of mathematics whilst engaged in the development of 21st century competencies. Wong and Kaur, in chapter 15, present the ACISK framework which is a tool for empowering mathematics learners to be self-directed. The framework may be used to i) to motivate students to self-direct their learning during mathematics instruction, and ii) to help students in their process of problem solving. In chapter 16, Thornton, exemplifies how students are empowered in their learning of mathematics through the Inquiry approach, so that students begin to work like mathematicians and be active contributors and concerned citizens, and not just passive learners. He shares with readers the fundamentals of the Australian reSolve: Mathematics by Inquiry Project, a mathematical task suitable for Years 7 or 8 students and describes professional resource for teachers of the project.

Cheng and Teong, in chapter 17, illustrate how self-regulated learners may be developed in the primary mathematics classroom. They describe the phases and processes of self-regulation adapted from a researched model and show through examples how these can be actualised in the classroom. Lastly, in chapter 18, Chan et al. highlight that a discursive
activity is an empowering activity as social interactions and sense-making through connecting, adding and correcting between pieces of knowledge provide the scope for learners to be empowered as active contributors and confident problem solvers in a collaborative setting. They draw on a group case study and discuss a modelling approach to empowering primary five students in learning mathematics through engaging a Model-Eliciting Activity (MEA).

6 Concluding Thoughts

Empowered learners of mathematics will be able to keep abreast of all future demands in a fast changing global landscape of today’s world. Therefore it is necessary for teachers to play their role in developing mathematically literate citizens of the world. The chapters in this yearbook provide readers and specifically classroom teachers with ideas on the why, what and how of empowering mathematics learners. Readers are urged to read the chapters carefully and try some of the ideas in their classrooms and convince themselves that these ideas offer a means to engage students in meaningful mathematical practices meant to develop the desired learning outcomes.

References


Chapter 2

Empowering Learning in an Algebra Class: The Case of Expansion and Factorisation

CHUA Boon Liang

Expanding products of binomials and factorising quadratic trinomials are two central topics in the algebra curriculum. Between the two, the latter is the one in which many students struggle to learn. Traditionally the teaching of expansion and factorisation is developed in a procedural way, often with too much reliance on teacher demonstration and explanation. Such a teacher-centred approach has to change if teaching is to move towards empowering students to be active learners engaged in discovering and generating new knowledge, and in taking charge of their own learning. Given the difficulties that students encounter with factorisation of quadratic trinomials, this chapter illustrates a teaching unit designed to help teachers develop the topics of expansion and factorisation in a more student-centred way, and with greater reliance on student comprehension and explanation. Not only do the activities in the teaching unit hope to support students in developing the expanding and factorising skills and in appreciating the inverse relationship between expansion and factorisation, but also to encourage them to assume greater responsibility in their own learning and to become self-directed and independent learners.

1 Introduction

Traditionally the teaching of mathematical procedural skills is carried out with the teachers showing worked examples, followed by assigning
practice question and then providing feedback to students. This approach of teaching tends to place emphasis on the teachers imparting knowledge to the students rather than on the students exploring and acquiring knowledge for themselves. While such a teacher-centred approach is useful to convey knowledge, it does not, however, seem to promote much critical thinking in students. To develop a thinking student, the teaching approach needs to be more student-centred. In other words, the focus of teaching has to shift from just the teachers imparting knowledge to allowing the students to engage in discovering new knowledge and take responsibility of their own learning.

This chapter sets out to demonstrate how mathematics teachers can move towards planning and creating teaching sessions that are more student-centred to empower students in order to improve their learning of mathematics. The demonstration of teaching sessions is framed in the context of teaching the expansion of a product of two linear binomials and the factorisation of quadratic trinomials. These mathematical topics are selected for two reasons: (i) expanding linear binomials and factorising quadratic trinomials are two key topics in the algebra curriculum, with the latter being the one in which many secondary school students struggle with and find it hard to learn, and (ii) informal personal chats with secondary school mathematics teachers have revealed that expansion and factorisation are commonly taught in a procedural manner for students to then model after, thus providing good grounds for illustrating how the two mathematical topics can be developed using a more student-centred teaching approach to promote deeper learning. So this chapter aims to offer some useful teaching ideas to encourage mathematics teachers to consider empowering their students to discover and construct new knowledge on their own. The chapter is structured along the following strands of work:

(a) a perspective of what empowerment in education means and why it is important for student learning,

(b) the Singapore secondary mathematics syllabus on expansion and factorisation,
Empowering Learning in an Algebra Class

2 What is Empowerment?

Empowerment is a management practice of sharing information and resources with workers, as well as encouraging them to not only take initiative and assume greater responsibility and authority, but also engage in the decision-making process in order to improve service and performance (Cartwright, 2002). This practice is seen as a process of tapping on people’s strengths and unleashing their full potential for them to achieve and grow in their careers (Yeap, Myint, Lim, & Low, 2005).

Applying the practice of empowerment in education, teachers too can help students grow and succeed in their own learning by nurturing them with certain crucial competencies such as critical thinking, problem solving, and decision making. Engaging the students actively in, for instance, critical thinking helps to sharpen their evaluative and analytical skills, as well as pushes and trains them to be discerning in judgment and in making a decision (Chua, 2016). This critical thinking skill will come in particularly handy in the case of having to justify a mathematical claim at any point of learning. The students, on their own, will be able to analyse the mathematical context from which the claim is established and then to construct logical arguments to convince either themselves or others about the veracity of the mathematical claim. From this perspective, empowerment can be seen as a useful teaching approach for preparing the students to become self-directed, independent and lifelong learners within a nurturing environment (Yeap et al., 2005). These student attributes are also aptly aligned with the framework for 21st century competencies and student outcomes (Ministry of Education (Singapore), 2014).
3 Expansion and Factorisation in the Singapore Secondary Mathematics Syllabus

The concepts of expansion and factorisation of algebraic expressions are introduced in Secondary One and Two in Singapore. The contents covered in the mathematics syllabus are presented in Table 1.

Table 1
Expansion and factorisation contents in the mathematics syllabus

<table>
<thead>
<tr>
<th>Level</th>
<th>Content</th>
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</table>
| Secondary One | • Expand and simplify algebraic expressions such as $3(a + 2)$, $5p(2 - r)$, and $-2(3x - 5y) + 4x$  
• Factorise algebraic expressions involving common factors such as $15x + 20y$, and $6ab - 3bc + 15b$  
| Secondary Two | • Expand a product of algebraic expressions such as $(c + 2)(c + 3)$, $(2 + 3x)(2x - 1)$, and $(x - 3y)(2x + y)$  
• Factorise quadratic expressions of the form $ax^2 + bx + c$  
• Use of quadratic identities such as $(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$, and $(a + b)(a - b) = a^2 - b^2$ |

To expand a product of algebraic expressions, mathematics teachers are advised to use a variety of methods such as the concrete approach using AlgeDiscs™, the multiplication-frame (MF) method and the symbolic approach by applying the distributive law of multiplication (e.g., the widely known FOIL method, which many mathematics teachers in Singapore called the Rainbow method) to provide different learning experiences to students. Figure 1 shows the expansion of $(x + 2)(x + 3)$ using the MF method (see Figure 1(a)) and the FOIL method (see Figure 1(b)).
The factorisation method depends on the type of algebraic expressions given. For algebraic expressions whose terms contain common factors (e.g., $6ab - 3bc + 15b$), students are taught the symbolic approach by applying the distributive law in reverse. This involves identifying and then extracting the highest common factor amongst the terms. For the factorisation of quadratic trinomials, teachers normally use multi-modal representations and different methods such as trial and error, cross multiplication, and multiplication frame. In the Singapore secondary mathematics syllabus, the use of AlgeDiscs™ and the MF method is highly advocated (Ministry of Education (Singapore), 2012). The MF method for factorisation is well known online as the Box method.

4 The Teaching Unit on Expansion and Factorisation

This section begins with a description of the pedagogical and task design considerations that are taken into account in the planning of the teaching unit, then followed by a demonstration of how these considerations are manifested in the lessons. The teaching unit is designed for Secondary Two students in the Express course. It deals with quadratic expressions, beginning with the expansion of a product of two linear binomials and then followed by the factorisation of quadratic trinomials. The learning objectives to be achieved in this teaching unit are as follows:

(a) recognise factorisation of an algebraic expression as a reverse process of expansion,
(b) expand the product of two linear algebraic expressions,
(c) factorise $ax^2 + bx + c$ using the MF method.
4.1 Pedagogical and task design considerations

In the teaching of mathematics, making connections among different mathematical ideas is of crucial importance because it helps students to make sense of what they learn in mathematics. So when designing the teaching unit on expansion and factorisation, the hierarchical nature of mathematical concepts and their inter-relationships are taken into consideration. This means that the tasks must build on the prior knowledge that students bring to the lesson and connect new ideas to their existing schema. Thus students will have to realise that the procedure for expanding two linear binomials is simply an extension of what they had learnt about expanding algebraic expressions in Secondary One, and that factorising quadratic trinomials requires undoing the binomial expansion procedure.

Students can appreciate the meaning of mathematical concepts better through using different modes of representation of the mathematical concepts (e.g., Sierpinska, 1992; Lesh, Behr, & Post, 1987). This teaching unit utilises multi-modal representations to make the concepts of expansion and factorisation more meaningful to students. By linking concepts and procedures, students will develop a deeper and more conceptual understanding of what expansion and factorisation are and why the procedures work the way they work.

As mentioned previously, many methods are available to expand binomials and factorise trinomials but not all may make sense to students. A case in point is the cross-multiplication method for factorisation, as illustrated in Figure 2 using the example of $x^2 + 8x + 12$. The first step is to determine a pair of factors for $x^2$ and for 12. Next, multiply a factor of $x^2$ with a factor of 12 to find its product (i.e., follow the solid arrow in Figure 2) and then repeat the step for the remaining pair of factor of $x^2$ and factor of 12 (i.e., follow the broken arrow). Finally, add up the two products to check that it matches the linear term of the trinomial. As can be seen, the method’s name derives from the cross-multiplication of factors.
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Figure 2. Cross-multiplication method

It is important that teachers know which method is efficient or is readily understood by students, and then decide on an appropriate method to teach. This teaching unit employs only the MF method for both expansion and factorisation. A prime reason for choosing to use the same method, rather than two different methods such as the FOIL method for expansion and the cross-multiplication method for factorisation, is to highlight and emphasise the relationship between expansion and factorisation. If students were to appreciate that factorisation is the reverse process of expansion, it is then crucial for them to realise that the method used for expansion should be reversible to work for factorisation too.

Student participation in class fosters active learning, which can in turn make learning effective. I think students will begin to feel empowered and motivated to learn more when they are actively engaged in their own learning. So to infuse the idea of empowerment into the teaching unit, the learning experiences must include opportunities for students to be engaged in active question-and-answer as well as discussion during the lessons, and to be involved in discovery activity. This approach supports the Singapore Ministry of Education’s emphasis on providing students with enriching learning experiences to help them “develop a deep understanding mathematical concepts, and to make sense of various mathematical ideas” (Ministry of Education (Singapore), 2012, p. 15). So students are not only taught to perform an algorithm procedurally to master expansion and factorisation skills but are also engaged in pattern recognition, reasoning and justification. The teaching unit will therefore include tasks that allow them to look for regularity in the expansion of two binomials using the MF method, as well as to analyse the expansion procedure to figure out how to reverse this procedure to factorise trinomials. Once the teacher actively engages the students in their own learning and imparts competencies like problem solving and critical thinking, they will not only start to feel
Empowering Mathematics Learners

empowered and confident, but also become responsible for their own learning.

Table 2 presents the pedagogical and task design considerations for the teaching unit in greater detail. It is hoped that the questions shown in the table will give teachers clearer ideas into the design of the teaching unit.

Table 2
Details of the pedagogical and task design considerations

<table>
<thead>
<tr>
<th>Concepts</th>
<th>Prior knowledge: algebraic expression, linear algebraic expression</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>What's new: how to explain expand, factorise and quadratic expression.</td>
</tr>
<tr>
<td></td>
<td>• What examples and non-examples to use in the concept explanation? For instance, examples of quadratic expressions include ( x^2 + 3x + 2 ), ( \frac{1}{2}x - 3x^2 ), and ( 5x^2 - 4 ) whereas ( 2x - 4 ), ( x^2 + 5x^3 ), and ( x^2 + \frac{2}{x} ) are three non-examples.</td>
</tr>
<tr>
<td></td>
<td>• What are the different representations of the various concepts and which ones to use? For instance, expansion of an algebraic expression can be introduced using algetiles or AlgeDiscs™, followed by using a diagram of a larger rectangle divided into smaller rectangles whose individual areas are established and then summing up the areas using mathematical symbols.</td>
</tr>
<tr>
<td></td>
<td>• How to illustrate the expansion-factorisation relationship using the MF method?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Skills</th>
<th>Prior knowledge: simply and factorise linear algebraic expression,</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>What’s new: expand the product of two linear binomials, factorise quadratic trinomials.</td>
</tr>
<tr>
<td></td>
<td>• How to connect expansion of the product of two linear binomials to that of a monomial and a binomial?</td>
</tr>
<tr>
<td></td>
<td>• How to relate factorisation of linear algebraic expression to that of quadratic expression?</td>
</tr>
<tr>
<td></td>
<td>• Which method of expansion and factorisation will best illuminate the relationship between them?</td>
</tr>
<tr>
<td></td>
<td>• How to introduce expansion and factorisation using the MF method?</td>
</tr>
<tr>
<td></td>
<td>• What instructional activity to design that will encourage discovery and critical thinking?</td>
</tr>
</tbody>
</table>
4.2 Development of the teaching unit

Part 1  At the start of the lesson, it is crucial that mathematics teachers look into the students’ background knowledge. The information gathered will help them make decisions about the development of instructional tasks and the choice of teaching approaches. For the teaching unit on expansion and factorisation, consider giving students a few algebraic expressions in the form of a product of a monomial and a linear binomial (e.g., \(3(2x + 1)\), \(x(4 + 3y)\) and \(-4x(2y + z)\)) to see what method they will use to expand them and whether they will perform the expansion correctly. The students should be familiar with the use of AlgeDiscs™ and the distributive law. Since the MF method is not initiated in the Secondary One mathematics syllabus but will be introduced later in this teaching unit, teachers may want to illustrate how the method works for the product of a monomial and a linear binomial and then get the students to compare and explain how it resembles the concrete approach using the AlgeDiscs™. This learning experience allows the students to be engaged in making observations, thinking and justification.

Following the assessment of the students’ expansion skill, teachers can next give the students new algebraic expressions containing common factors to test their factorisation skill. The students should be able to apply the distributive law in reverse to factorise those expressions. To pique their curiosity, ask them to figure out and explain how the MF method can be employed to do the factorisation. As the students are unlikely to think of undoing the expansion procedure using that method, a teacher-led inquiry can then ensue to review what factorising an algebraic expression means and how it is related to expansion. A manifestation of how a mathematics teacher can lead students to factorise an expression using the MF method is presented in Figure 3. After some discussions, it is hoped that the students will realise that undoing the method for expansion leads to factorisation.
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<table>
<thead>
<tr>
<th>Teacher prompts</th>
<th>To be presented on whiteboard</th>
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<tbody>
<tr>
<td>Task: Factorise $15x + 20y$.</td>
<td>Show the $2 \times 3$ grid.</td>
</tr>
<tr>
<td>1. Where do you write $15x$ and $20y$ in the grid?</td>
<td><img src="image" alt="2 x 3 grid" /></td>
</tr>
<tr>
<td>2. There are now three blanks in the grid. Which one do you fill in first? (Expected: 2nd row blank)</td>
<td><img src="image" alt="Grid with 15x and 20y" /></td>
</tr>
<tr>
<td>3. How do you find the term in this blank?</td>
<td></td>
</tr>
<tr>
<td>4. How is this term related to $15x$ and $20y$? (Expected: the highest common factor of $15x$ and $20y$)</td>
<td></td>
</tr>
<tr>
<td>5. What do you now write in the remaining blanks?</td>
<td><img src="image" alt="Grid with 5 and 15x, 20y" /></td>
</tr>
<tr>
<td>6. How do you obtain your answers?</td>
<td></td>
</tr>
<tr>
<td>7. How do you know the factors are correct?</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3. Factorisation by the MF method

Part 2 The expansion of two linear binomials using the MF method will be introduced here. Before demonstrating the procedure, begin with a discussion of the new key mathematical phrase, *quadratic algebraic expression*, to familiarise students with the vocabulary used in the lessons. Then get the students to identify the quadratic algebraic expressions from a list of examples.

The expansion of two linear binomials can be introduced using concrete algebra tiles. The use of algebra tiles helps students to appreciate the geometrical meaning of the algebraic expressions involved. For instance, construct a rectangle of sides $(x + 2)$ and $(x + 3)$ on the board,
and ask the students to write down its area, in terms of $x$, in two different forms: $(x + 2)(x + 3)$ and $x^2 + 5x + 6$. Next, encourage them to examine these two different-looking expressions and infer a relationship between them. The students should be able to accomplish this task correctly quite effortlessly. Despite being a very simple task, it allows the teachers to reiterate the equivalence of algebraic expressions and to reinforce the area concept of expansion.

The lesson can progress to build on the concrete approach with the MF method. The latter offers a similar but simpler way than the concrete approach to expand a product of two binomials. One advantage of the MF method over the concrete approach is that it circumvents the tricky issue when negative terms are involved. Consider the expansion $(x + 2)(x - 1)$ using algebra tiles. One configuration that mathematics teachers may construct for the expansion is shown in Figure 4a. This configuration reveals the incorrect way of using algebra tiles to multiply the pair of binomials. This is because $(x - 1)$ representing the length of a rectangle cannot be longer than $x$, and the negative areas of $-x$ and $-2$, represented by the shaded tiles, also do not make sense to students. The correct way to use the algebra tiles is to place the shaded tiles within the unshaded tiles to indicate a decrease in length (see Figure 4b). Even for this second configuration, cognitive conflict can arise when the physical size of the rectangle does not match the algebraic expression representing its area. Students can get confused and wonder why the rectangle looks visually larger than its area, which is given by $x^2 + 2x - x - 2$.

![Figure 4. Representation of $(x + 2)(x - 1)$ using algebra tiles](image)
For this phase of the learning on expansion, the teacher can adopt the question-and-answer approach here. Asking questions can stimulate students’ thinking and check their understanding and reasoning. This is why it is recommended over simply instructing them on what to do. To illustrate the procedures, consider using the same algebraic expression $(x + 2)(x + 3)$ as in the concrete approach. Ask students, for instance, “What is the first step that you do? Where do you write $x + 2$ and $x + 3$ in the grid? How do you now find each of the blanks in the grid?” The teacher fills in what the students respond into the grid. After completing the grid, ask the students to compare with the concrete approach and then explain what the term in each blank represents. As well, ask them how they will establish the terms of the expanded form of $(x + 2)(x + 3)$. The teacher can then invite individual students to reiterate the steps, at the same time displaying the steps on the board (see Figure 5). Depending on student learning, a few more examples may be needed. Subsequently, the students will practise the steps on other examples individually. The examples can include binomials involving subtraction. As they work on the examples, encourage them to whisper the steps to a peer or even to themselves silently. Vygotsky (1987) noticed that such private speech can act as a cognitive tool to help learners overcome the obstacles that they encounter along the way when working on a task, and boost their thinking as well as understanding.

After the individual practice, encourage the students to study carefully the four terms that are derived in all examples and practice questions to look for patterns (see Figure 6). Depending on what the students notice, there could be a range of student responses for this learning activity. The responses are likely to revolve around surface features: for instance, there are two like terms amongst the four terms, the quadratic term is always located in the top left corner, and the quadratic and constant terms are in opposite corners and so are the two linear terms.
Expand \((x + 2)(x + 3)\).

**Step 1:** Write the terms in the first binomial in the first row, and those in the second binomial in the first column. The four boxes at the lower right are each assigned a number 1 to 4 in a clockwise direction starting with the top left corner.

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<thead>
<tr>
<th></th>
<th>(x)</th>
<th>2</th>
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<tbody>
<tr>
<td>(x)</td>
<td>Box 1</td>
<td>Box 2</td>
</tr>
<tr>
<td>3</td>
<td>Box 4</td>
<td>Box 3</td>
</tr>
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</table>

**Step 2:**
Box 1 contains the product of \(x\) and \(x\). Generate the products in Box 2 to Box 4.

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<tbody>
<tr>
<td>(x)</td>
<td>(x^2)</td>
<td>2(x)</td>
</tr>
<tr>
<td>3</td>
<td>3(x)</td>
<td>6</td>
</tr>
</tbody>
</table>

**Step 3:** Add up the terms in the four boxes to obtain the trinomial.

- Box 1: \(x^2\)
- Box 2 and Box 4: \(2x + 3x = 5x\)
- Box 3: 6

\((x + 2)(x + 3) = x^2 + 5x + 6\)

*Figure 5. Steps for expansion by the MF method*

The last surface feature mentioned previously is of particular importance because it leads to a key mathematical observation for factorising quadratic trinomials: that is, *the product of the two linear terms is equal to the product of the quadratic and constant terms*. This mathematical feature is, however, not easy to spot and can pose a huge challenge to students. Therefore the teacher may have to draw the students’ attention to certain focal attributes in this learning activity for them to discover it. One way to do it is to get them to determine the products of the two pairs of opposite corners for all the examples and practice questions that they have already worked out (see Figure 6), followed by observing and inferring a relationship between the two products. It is hoped that they will then notice the equivalence of the product of the linear terms and the product of the quadratic and constant terms. In the spirit of inquiry learning, provide learning experiences that
foster mathematical reasoning in the classroom. One task that the teachers can require students to do is to justify why the mathematical feature holds. This task can deepen the students’ appreciation of the MF method.

**Figure 6. Products of opposite corners**

**Part 3** The factorisation of quadratic trinomials using the MF method will be dealt with here. Having previously learnt in Part 1 that undoing the MF method for expansion leads to factorisation, the students can work in pairs to figure out how to factorise the quadratic trinomial $x^2 + 8x + 12$
using the MF method. This first factorisation activity with no teacher
guidance or support can be tough for students, so it may be necessary to
provide sufficient time for them to explore.

After students have attempted the exploratory activity, the teacher
could invite a few successful ones to share how they performed the
factorisation correctly with the class. This opportunity to present their
work to the entire class promotes effective and precise communication of
ideas, as well as trains the students to become confident learners, a
desirable student outcome as delineated by the Singapore 21st century
competencies framework (Ministry of Education (Singapore), 2014). But
if no one is successful in factorising $x^2 + 8x + 12$ using the MF method,
the teacher can then prompt the students to better engage them in the
learning process. For instance, ask questions such as “which of the three
terms in $x^2 + 8x + 12$ can be placed in the grid” and “in which box are
they placed” to get the students to place the quadratic and constant terms
correctly in opposite corners. Ask them next how to fill in the remaining
linear term $8x$ in the two boxes that are left in the grid. Although this task
can be tricky for students, a few should be able to recognise the need to
decompose $8x$ into the two like terms, $2x$ and $6x$. Crucially, opportunities
should also be provided for them to justify how they decided on those two
like terms.

In the case when no one is able to explain how to generate the like
terms, this is where teacher guidance is needed. There are two ways to
decompose the linear term. The first method is to make use of the result
established in Part 2: that is, find the product of the quadratic and constant
terms, followed by determining linear factor pairs of this product to see
which pair adds up to $8x$. The other method is to establish pairs of linear
like terms that add up to $8x$, followed by finding the pair that when
multiplied together produces $12x^2$. An important question that the teacher
may want to discuss with the students concerns the arrangement of the like
terms along the opposite corners: that is, does it matter whether a particular
like term occupies the top corner or the bottom corner? After the class
discussion, the students should come to the conclusion that the position is
unimportant. Subsequently, guide the students to find the highest common
factor of the two terms in each row and each column in the grid to derive
the two linear binomials, as well as emphasise the checking of answer at
Empowering Mathematics Learners

Figure 7 outlines the steps of factorisation by the MF method. Individual student practice on a variety of trinomials then follows.

After this student practice, a worthwhile classroom activity is to engage students in a discussion to share efficient ways of selecting the linear factor pair in Step 2a in Figure 7. The purpose of this activity is to draw their attention to realise that if the product established in Step 1b is positive and the linear term in the given quadratic trinomial is negative, then the factor pair should be both negative. One benefit of noticing this mathematical feature is that it saves time from having to consider factor pairs that are both positive and cuts the chance of missing out the factor pair.

The MF method in this chapter differs from the one described in Singapore mathematics textbooks. As outlined in Figure 8 using the example of \( x^2 + 8x + 12 \), the textbook method commences with locating the quadratic and constant terms in the appropriate positions, then establishing their factors, followed by checking which case yields the linear term in the given quadratic trinomial. A drawback of this textbook method is that it does not correspond to undoing the expansion process using the MF method (see Figure 5). This problem may create a disconnect in the inverse relationship between factorisation and expansion.

<table>
<thead>
<tr>
<th>Factorise ( x^2 + 8x + 12 ).</th>
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<tbody>
<tr>
<td><strong>Step 1a:</strong> Write the quadratic and constant terms in opposite corners.</td>
</tr>
<tr>
<td><strong>Step 1b:</strong> Establish the product of these two terms. The product is ( 12x^2 ).</td>
</tr>
<tr>
<td><strong>Step 2a:</strong> Decompose the product found in Step 1b into linear factor pair, and look for the pair that adds up to ( 8x ). (Alternatively, decompose the linear term ( 8x ) into pairs of like terms, and look for the pair that when multiplied together gives ( 12x^2 ).)</td>
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<tr>
<td><strong>Step 2b:</strong> Write down the two terms in the remaining corners.</td>
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</table>
Step 3a: Consider the two terms in the middle row. That is, $x^2$ and $2x$ in this example. Find the highest common factor (HCF) between them and write it down on the far left.

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<td>$2x$</td>
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<td>$6x$</td>
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Step 3b: Divide each of the two terms by their HCF and write down the results in the top row. For instance, the results of dividing $x^2$ and $2x$ by $x$ are $x$ and $2$ respectively, thus producing the factor $(x + 2)$. (Note: This step should be familiar to students because it corresponds to extracting common factors, which is a skill developed previously.)

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<td>$6x$</td>
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Step 3c: Consider the two terms in the bottom row. Find the HCF between them and write it down on the far left.

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Step 3d: With the establishment of the factors, check that the product of the HCF obtained in Step 3c and the factor obtained in Step 3b matches the terms in the bottom row. For instance, multiply 6 and $x$ gives $6x$ and multiply 6 and 2 gives 12.

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<td>$6x$</td>
<td>$12$</td>
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Step 4a: Express the quadratic trinomial as a product of the two binomials established in Step 3d.

\[ x^2 + 8x + 12 = (x + 2)(x + 6) \]

Step 4b: Verify your answer by expanding the two linear binomials.

Figure 7. Steps for factorisation by the MF method
On the other hand, the MF method described in Figure 7 works very much like undoing the expansion process outlined in Figure 5. In particular, Step 3a to Step 3d in Figure 7 build on the students’ prior knowledge of extracting common factors. This is one prime example illustrating how an existing schema can be recalled to develop a new concept. Presently it has not yet been established empirically whether this method is more effective than the textbook method, but anecdotal evidence from my personal experience working with a few Secondary Two students in the Normal (Academic) course suggests that the non-textbook method holds promise. The students, all from the same class, were taught the textbook MF method by their Mathematics teacher. They had to attend an enrichment workshop conducted by me because they failed their year-end mathematics examination. One of the tasks assigned to the students attending the workshop required them to factorise $6x^2 - 17x + 12$. After a few futile attempts of trial and error using the textbook method, the students became frustrated and gave up. The non-textbook MF method was subsequently introduced to them as an alternative method. After some practice on other quadratic trinomials, the students revisited $6x^2 - 17x + 12$ and successfully factorised it using the non-textbook method. Verbal feedback from these students was positive: for instance, the non-textbook MF method is easier to follow and more systematic than the textbook method. Further research will need to be undertaken to examine the effectiveness of the MF method described in this chapter.

\[
\begin{array}{c|c|c}
\times & x^2 & 12 \\
\hline
x^2 & & \\
\end{array}
\]

Factorise $x^2 + 8x + 12$.

**Step 1:** Write the quadratic term in the top-left corner and the constant term in the bottom-right corner in the grid.
**Step 2:** Consider the factors of $x^2$ and 12. Write them down in the appropriate column and row. For instance, the factors of $x^2$ can be written as $x^2 = x \times x$. For factors of 12, the possibilities are $1 \times 12$, $2 \times 6$, and $3 \times 4$. Hence, there are three cases to consider.

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<td>$x^2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$x$</td>
<td>$x^2$</td>
<td>$12$</td>
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Use the factors to check if the result matches the given quadratic expression.

**Step 3:** Multiply the factors to find the linear terms and check if the result matches the given quadratic expression.

Of the three cases, Case 1 and Case 3 are rejected as the sum of the linear terms does not match the linear term in the given quadratic expression. Case 2 is the correct one.

Hence $x^2 + 8x + 12 = (x + 2)(x + 6)$.

**Case 1:**

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<td>$x$</td>
<td>$x^2$</td>
<td>$12x$</td>
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$x + 12x \neq 8x$

**Case 2:**

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<td>$x$</td>
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$2x + 6x = 8x$

**Case 3:**

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<td>$3x$</td>
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<tr>
<td>$x$</td>
<td>$x^2$</td>
<td>$4x$</td>
</tr>
<tr>
<td>$x$</td>
<td>$x^2$</td>
<td>$12$</td>
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$3x + 4x \neq 8x$

*Figure 8. The MF method in Singapore mathematics textbook*
5 Concluding Remarks

Mathematical skills such as algebraic manipulation like expansion and factorisation tends to be taught in a procedural manner, placing little emphasis on student inquiry and explanation. This way of learning skills deprives students of an enriching learning experience in which they discover and create new knowledge for themselves, as well as engage in mathematical reasoning and justification. The above sections show how the mathematical skills of expansion and factorisation can be developed in a more student-centred manner through incorporating the idea of empowerment.

The practice of empowerment is an investment in students, tapping on their potential for them to achieve and grow in their learning. An empowering learning process enables the students to think critically and take responsibility of their own learning. Students can develop confidence in applying their mathematical knowledge and reasoning through learning activities that foster critical thinking (Wong, 2016). With confidence, students will feel motivated to learn independently, thus cultivating self-directed learning. All these are important competencies to prepare the students to work effectively in the global landscape of the 21st century.

References

Empowering Learning in an Algebra Class


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Chapter 3

Facilitating Students’ Mathematical Noticing

TAN Liang Soon      HANG Kim Hoo

Much as teacher mathematical noticing has been a key component in mathematics teaching expertise, students’ mathematical reasoning in the learning, doing and using of mathematics can be influenced by their mathematical noticing. In this chapter, we describe three aspects of the mathematics teacher classroom practices that may contribute to students’ mathematical noticing, namely cognitive offloading, re-orienting and instantiation. These were revealed in a case study involving an A-level Further Mathematics classroom in Singapore. Lesson segments are discussed to illustrate these aspects of the mathematics teacher classroom practices.

1 Introduction

It is generally believed that students’ mathematics performance is dependent on their mathematical reasoning abilities. But students’ mathematical reasoning can be influenced by their mathematical noticing of important aspects of the problem situations. Students’ mathematical noticing is about giving attention to particular aspects of the problem situation amid competing sources of information.

In an example to illustrate students’ mathematical noticing, the task of finding the composite function $f^{12}(x)$ from its given function $f(x) = x + 2\sqrt{x} + 1, x \geq 0$ was presented to two Junior College (JC) one (Year 11) students. Different mathematical reasoning trajectories were engendered, depending on what the students noticed or did not notice. One student just noticed the task of finding the composite function $f^{12}(x)$ and
plunged straight into that arduous process. Yet the other student noticed
the completed square structure of the function \( f(x) \) and reasoned that:

\[
f(x) = (\sqrt{x} + 1)^2, \ x \geq 0
\]

\[
f^2(x) = \left( (\sqrt{x} + 1)^2 + 1 \right) = (\sqrt{x} + 2)^2
\]

\[
\therefore f^{12}(x) = (\sqrt{x} + 12)^2.
\]

As another example, secondary three (Year 9) students were asked to
find the area of triangle \( QRS \) in the semi-circle of radius \( r \) cm where the
lengths of arcs \( PQ, QR \) and \( RS \) are equal (see Figure 1). Some students
would go through a longer solution route by subtracting the areas of
triangle \( OQS \) and segments \( QR \) and \( RS \) from the area of sector \( OQS \).
For those students who could notice triangles \( OQS \) and \( QRS \) are
congruent, their mathematical reasoning process would have been very
different.

\[\text{Figure 1. A circular measure problem}\]

The re-introduction of Further Mathematics at the A-Levels in
Singapore is a positive development in providing better opportunities for
interested or talented students to learn, do and use mathematics more
Facilitating Students’ Mathematical Noticing

deeply. The classroom practice of the teacher in implementing the Further Mathematics curriculum is the subject of research in this study. Specifically, we want to examine how the mathematics teacher can facilitate students’ mathematical noticing in a Further Mathematics classroom. This study addressed the following research question:

What are some aspects of a mathematics teacher’s classroom practice that facilitated students’ mathematical noticing?

In the first part of this chapter, we provide background about noticing, in terms of its influence from cognitive psychological factors. We then examine how particular aspects of the teacher’s classroom practice have the potential to foster students’ mathematical noticing of important aspects of the problem situation. The findings from this study can provide teachers with some suggested pedagogical approaches to engage students in mathematical noticing, which can in turn contribute to the development of students’ mathematical reasoning competencies.

2 Noticing

The way novel problem situations are perceived can influence to a large extent one’s treatment of the situations. Marton and Booth (1997) had pointed to a considerable body of research that demonstrated this phenomenon. That is, “What you do not notice, you cannot act upon” (Mason, 2002, p.7). Indeed, what differentiates experts from novices is the way they see novel situations in their field of expertise (Bransford, Brown & Cocking, 2000). The idea of intentional noticing as proposed by Mason (2002) was explored in numerous studies involving mathematics teachers’ noticing and its consequences for their pedagogical practices (Jacobs, Lamb & Philipp 2010; Sherin, 2001, 2007; Sherin & van Es, 2005, 2009; Star & Strickland 2008; van Es & Sherin, 2002, 2008). The focus of teacher noticing in these studies ranged from the classroom environment, the mathematical tasks and content, communication and students’ mathematical thinking. Their findings highlighted the dependency of teacher noticing on their experiences, educational beliefs and cultural
backgrounds and that teachers’ experiences can be scaffolded to notice in particular ways.

While extensive work had been done in the area of mathematics teacher noticing over the past decades, reports of attempts to examine students’ noticing are only beginning to appear in the research literature. Lobato, Hohensee and Rhodehamel (2013) investigated the influence of students’ noticing of the problem situations on their mathematical reasoning process. They examined students’ noticing in two middle grades mathematics classes and its effects on subsequent mathematical reasoning tasks. In maintaining the same content goals, they noted that students’ mathematical noticing can be distributed across discourse practices, mathematical tasks and learning activities. Other cognitive research studies had examined how learning gains in the subject domains of reading, Economics, and Chinese Language are related to key noticing elements of attentional treatments, contrasting cases, and systematic variation (Chik & Marton, 2010; Marton & Pang, 2006; McCandliss, Beck, Sandak, & Perfetti, 2003).

As can be seen from the research literature, both cognitive-psychological and social-cultural factors can form the basis for students’ mathematical noticing. We shall limit our theoretical lens for this study to that of the cognitive psychology perspectives. In their study of how one attends to stimuli in the environments, cognitive scientists had identified three attentional functions of alerting, orienting and executive control (Fan, McCandliss, Sommer, Raz, & Posner, 2002). Alerting is about activating different levels of alertness in response to the types of tasks. Orienting is to direct attention to the cued location or stimulus. Executive control is activated to resolve multiple responses in tasks involving cognitive conflict and other forms of mental effort. The brain centres responsible for each of these attentional functions are different. Although the act of noticing a given phenomenon is more aligned to the executive control function (Lobato, Hohensee & Rhodehamel, 2013), its process can comprise a composite of these three types of attentions that may be functionally independent (Fan et al., 2002).

Another important concept related to students’ noticing is inattentional blindness, a term coined by psychologists to describe our focusing limitations (Simons, 2000). Studies have found that observers fail
to notice salient objects when they are focused on some other object or event (in a state of attention). A well-known study that illustrates this phenomenon showed that a large percentage of observers who were focused on counting the number of passes in a video of basketball game did not notice the arrival of a man in a gorilla suit (Simons, 2000). In the introductory example task of finding the composite function \( f^{12}(x) \), the student who failed to notice the completed square structure of the function \( f(x) \) could likewise have been limited by his focusing capacity in other competing ideas from composite functions.

Students might be advised to try to notice the structure of the function \( f(x) \). But students’ ability to notice the completed square structure would depend on their past experiences with variations of completed square expressions. It will be difficult for students to notice the completed square structure if they had only seen the standard form \( x^2 + 2x + 1 \). The idea that students’ noticing is a function of their experienced variation and invariance was discussed in detail by Marton, Tsui, Chik, Ko, and Lo (2004). Four conditions were deemed necessary for students’ noticing, they are namely, contrast, generalization, separation and fusion (Marton et al., 2004).

The students’ experience with contrasting objects can serve as a basis for comparison with the particular aspect in a problem situation they need to notice. Say to notice red colour, one must experience non-red colour. In addition, one must have experience with variants of red, for example, red toys, red lips, red pens, and so on. For such generalized experience to be effective, the idea of “redness” must first be varied separately from the other aspects, say by keeping the quantity of the objects invariant. It is then important to fuse these two aspects through the simultaneous variation of the appearances of “redness” and its quantity. For instance, students will experience two pairs of red lips, five red toys, and seven red pens. An examination of this type of fusion was carried out by Marton and Pang (2006). The elasticity ideas of demand and supply were first separated and then fused in a group of economics classes that were involved in a learning study. These ideas were only discussed separately in the other group of lesson study classes. The findings showed that the relationship between the relative elasticity of demand and supply and the distribution of tax
burden was better understood among the experimental group of students in the learning study.

Recognizing the relations of parts within the wholes of particular problem situation is another important aspect of students’ noticing (Marton et al., 2004). A study was carried out to examine the ways in which university students comprehended a particular piece of text (Saljo, 1982). It was found that students who noticed the hierarchical structure of the text (in terms of its part-whole relationship between the sub-themes and its main themes) had better understanding of its main ideas as compared to students who only understood the text in a sequential way. In another study, Gu (1991) (cited in Marton et al., 2004) investigated students’ difficulties in noticing the part-whole structure of geometric figures. It was suggested that students’ experiences with the set of figures that had undergone a series of geometric transformations could be useful in noticing the simple figures from the complex figures.

Collectively, these findings suggest the multi-faceted and individual characteristics of students’ mathematical noticing. Such cognitive-psychological factors can have its implications on teachers’ classroom practices. Due considerations of these factors when designing and implementing lesson activities can contribute towards facilitating students’ noticing of important aspects of the mathematical problem situation. This in turn can help direct students’ mathematical reasoning process.

3 The Study

3.1 Setting and participants

The study reported in this chapter is part of a learning study whose goal was to bring about the development of students’ mathematical reasoning competencies in a H2 Further Mathematics classroom. The theoretical construct of students’ noticing was used to frame the learning study. While the learning study spanned the topics of recurrence relations and conics, the study reported in this chapter is focussed on students’ learning of conics. Traditionally, conics has been used as tools in mathematical problem solving.
Facilitating Students’ Mathematical Noticing

The aim of the study was to examine the aspects of a mathematics teacher’s classroom practice that facilitated students’ mathematical noticing. We investigated the preceding question by means of a case study. Case study is the preferred research method for answering ‘how’ questions which are more explanatory, and when the focus is on a contemporary phenomenon within a real life Secondary school context (Yin, 2003). Gillham (2000) defined case study as a unit of human activity embedded in the real world. In the present study, the human activity being examined is the teacher’s classroom practice in the teaching of conics.

The case selected for in-depth discussion was a male teacher who carried out a lesson for the teaching of conics in a H2 Further Mathematics class. The teacher had more than 15 years of experience teaching mathematics at the secondary and JC levels. The H2 Further Mathematics class comprised 9 male and 3 female JC one students. The didactical contract (Brousseau, 1997) that was established with the students focused on mathematical reasoning. The teaching approach employed by the teacher was one that elicits, challenge and refine students’ mathematical reasoning as they were engaged in the mathematical problem solving process.
3.2 The mathematics problem task and implementation

We used a problem task that required students to derive the equation of the director curve for an ellipse. The director curve of an ellipse is the locus formed by the points of intersection between two perpendicular tangents to the ellipse. The task is posed as follows:

1. (i) Show that, if the line $y = mx + c$ is a tangent to the ellipse $E$ with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $c^2 = b^2 + a^2m^2$.

   (ii) The locus of points at which two perpendicular tangents to $E$ meet is called its director curve. Use the result of part (i) to find a quadratic equation in $m$, with coefficients in terms of $a$, $b$, $x$, and $y$, and hence show that the director curve of $E$ is a circle, stating its centre and radius.

   (iii) Deduce the equation of the director curve of the circle with equation $x^2 + y^2 = r^2$.

The students find such loci construction and derivation problems in conics to be challenging.

The core mathematical ideas focused on notions of conics, gradient function, locus, and roots of quadratic equation. The sub-parts of the task were designed to elicit alternative ways to derive the director curve equation, which in turn requires students’ noticing of important aspects of the problem situation. The task was implemented over a period of two one-hour lessons. In the first lesson, the students worked in pairs to solve the problem. The teacher then facilitated students’ noticing of the important aspects of the problem situation in the second lesson. In this chapter, we report the results of the teacher’s practice while carrying out these two lessons.

3.3 Data collection and analysis

We video-recorded each lesson and made detailed field notes during the lesson. This was required to record and retain the teacher’s teaching
actions and his interactions with students. The students’ verbal responses and their gestures were also video-recorded in the process. The Swivl video camera system was thus used for this purpose. Notes on the researcher’s informal conversations with the teacher that occurred before and after the lessons were also made.

Relevant artefacts from students’ performance of the problem task were also collected. The students’ artefacts were in the form of their written solutions for the problem task. It was noted that students’ mathematical noticing were not directly observable. Therefore, in addition to students’ verbal responses and their gestures made during the lesson, these artefacts would be analysed to infer what students notice about the important aspects of the problem situation.

The transcripts of all the lesson videos, field notes and artefacts were prepared for use in the data analysis. The field notes and content logs of the lesson videos were used to select relevant segments of the lessons for data analysis. The method of data analysis was guided by our research question and was completed in two stages. In the first stage of analysis, the data collected was analysed through the constant comparative method (Glaser & Strauss, 1967). This method involved open-ended coding for the teacher’s classroom practice in facilitating students’ noticing of the important aspects of the director curve problem situation. We examined the teacher’s practice based on what had happened in the classroom. The codes were not pre-defined but derived from studying what actually happened. Simultaneous review and evaluation of all the units of meaning was done to refine the categories of codes. Samples codes, with their descriptions, and excerpts from the transcript of the lesson are shown in Table 1. In the second stage of analysis, these categories of codes were then clustered to derive themes that defined the important aspects of the teacher’s practice in facilitating students’ mathematical noticing. For example, the theme of instantiation emerged from the cluster of codes that included directing students’ attention and highlighting part-whole relationship.
<table>
<thead>
<tr>
<th>Category</th>
<th>Descriptor</th>
<th>Sample data of the teacher’s transcript</th>
</tr>
</thead>
<tbody>
<tr>
<td>Directing students’ attention</td>
<td>Highlighting important aspects of the problem situation</td>
<td>“For every point ((x, y)) on the (xy) plane, there is a given condition that this distance (between the point and the origin) is fixed. So I used this condition which leads me to derive the equation of the locus, which is a circle”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>“Now let me see, perpendicular! Hah! So there is an additional condition, this one (annotating right angle between the two perpendicular tangents in the diagram on the whiteboard) must be ninety degrees”</td>
</tr>
<tr>
<td>Highlighting part-whole relationship</td>
<td>Relates the elements of the problem situation to its solution process</td>
<td>“Now without looking at the question, how would you know it’s a circle? (Pause for a minute) Some mathematician had already solved it and this question helps to scaffold you step by step to arrive at this equation”</td>
</tr>
<tr>
<td>Getting students ready to notice</td>
<td>Ensures students are on the same page and ready for mathematical noticing</td>
<td>“If I leave the quadratic equation in this form, will you get full marks?” (Student said “no”)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>“So what must you do?” (Student replied that they need to expand)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>“Okay can you all expand?” (provided ten minutes for students to work out the quadratic equation in terms of (m), with coefficients in terms of (a, b, x) and (y))</td>
</tr>
</tbody>
</table>
4 Findings and Discussion

The focus in this study is on examining important aspects of the teacher’s classroom practice that contribute towards students’ mathematical noticing. The main findings are organized around the research question as discussed. We outline three aspects of the teacher’s classroom practice that facilitated students’ mathematical noticing, they are namely:

1. Cognitive offloading
2. Re-orienting
3. Instantiation

4.1 Cognitive offloading

The ellipse director curve problem task is more complex as compared to the standard locus of circle problem. In the standard locus of circle problem situation, the only variables are the $x$ and $y$ coordinates of the locus points, with the variable point moving under a fixed condition. In the director curve problem task situation, both the gradients of the tangents to the ellipse and its point of intersection are varying under the condition that the tangents are perpendicular to each other. This would demand more of the students’ attention to the competing sources of information in the problem situation. Consequently, this would warrant the use of learning activities to reduce the cognitive demand of the task and any inattentional blindness (Simons, 2000) that may arise.

During the first lesson, the students were engaged in pairs to work through the director problem task. By asking students to do that, the teacher was trying to achieve two things. First, he was allowing students to expend large amount of cognitive effort in the problem task space. In this way, the students would have been familiar with the problem situation when it comes to the second lesson where the teacher attempted to facilitate students’ mathematical noticing. This could enable students to overcome their focusing limitations (Simons, 2000) to attend more to the important aspects of the problem situation. Second, it allows the teacher to understand some of the students’ cognitive struggles. These understandings were instrumental to the teacher in designing appropriate
lesson activities to facilitate students’ mathematical noticing in the second lesson.

Another practice by the teacher to reduce the cognitive demand of students’ noticing was to get students ready to notice. Before the teacher transited to getting students to derive the director curve equation in sub-part (ii) of the problem task, he would ensure that all the students had worked out the quadratic equation in \( m \). The following excerpt illustrated this practice:

Teacher: *If I leave the quadratic equation in this form, will you get full marks?*

Student A: *No.*

Teacher: *So what must you do?*

Student B: *Need to expand.*

Teacher: *Okay can you all expand? (the teacher provided ten minutes for students to work out the quadratic equation in terms of \( m, y, a \) and \( b \))*

The teacher had deemed it important to break up the students’ noticing into manageable parts. In that, students who were still trying to work out the quadratic equation in \( m \) would unlikely be ready to notice the next important aspect of the problem situation.

At the initial stage of the second lesson, the teacher got students to experience variants of the standard loci, “Got the idea now? Let me give you another example.” The teacher proceeded to discuss locus of a circle with the students after discussing the locus of perpendicular bisector. In these discussions, the teacher emphasized how the locus equation were derived based on the given condition for which the variable point would move. Not only had the teacher’s variation practice allowed students to notice the given condition of the perpendicular tangents in the complex director problem task, it also helped to reduce its associated cognitive demand.
4.2 Re-orienting

During the first lesson, some students gave up trying to make sense of the solution process as suggested by the sub-parts of the problem task. However, they saw that the director curve equation could be derived by solving simultaneously for the points of intersection between the two perpendicular tangents to the ellipse $E$ (see Figure 2). It was noted that the students lacked the algebraic skills and fluency to see through the solution process.

![Figure 2. Student’s cognitive struggle in finding the director curve equation](image)

While the teacher could help students follow through this solution process (see Figure 3), he chose instead to re-direct students’ attention to particular aspects of the problem situation that correspond to the respective sub-parts of the problem task. He assured students he will “revisit this solution approach of finding the points of intersection of the two perpendicular tangents to the ellipse later” and asked them instead to “focus on the problem task.” He valued the solution process suggested by the problem task as it entails far richer learning experiences for students to notice the important aspects of the director curve problem situation.
\[ y = mx + \sqrt{a^2 m^2 + b^2} \]
\[ y - mx = \sqrt{a^2 m^2 + b^2} \quad \text{(1)} \]
\[ y = -\frac{1}{m} x + \sqrt{\frac{a^2}{m^2} + b^2} \]
\[ = -\frac{1}{m} x + \frac{\sqrt{a^2 + b^2 m^2}}{m} \]
\[ my + x = \sqrt{a^2 + b^2 m^2} \quad \text{(2)} \]

Squaring equations (1) and (2) respectively, we have
\[ y^2 - 2mxy + m^2 x^2 = a^2 m^2 + b^2 \quad \text{(3)} \]
\[ m^2 y^2 + 2mxy + x^2 = a^2 + b^2 m^2 \quad \text{(4)} \]

Adding equations (3) and (4), we have
\[ (m^2 + 1)(x^2 + y^2) = (m^2 + 1)(a^2 + b^2) \]
\[ x^2 + y^2 = a^2 + b^2 \]

The director curve is a circle with centre (0,0) and radius \( \sqrt{a^2 + b^2} \).

Figure 3. Alternative solution for the director curve equation

By identifying such a focus for noticing, in this case the gradients of the tangents \( m \) as a variable, as well as the given condition of the perpendicular tangents, students were able to attend less to their simultaneous solution approach, and more to how the variables and the conditions for the locus construction were related. The basis for this practice is to re-orient students’ attention to more relevant aspects of the problem situation, which creates the opportunity for students to reason mathematically in deriving the director curve equation.

One implication from these findings is that in mathematical problem solving tasks that often embed competing sources of information, students who are focusing on a variety of mathematical information that may not be relevant will need to be re-oriented to important aspects of the problem situation. The basis for this would be the cognitive psychology studies by Fan et al. (2002), whose discussion on the attentional function of orienting...
could constitute an important aspect in the process of mathematical noticing.

4.3 Instantiation

In the process of working through sub-part (ii) of the director problem task, the students had difficulties understanding why is it that they could formulate a quadratic equation in \( m \), with coefficients in terms of \( a, b, x \) and \( y \). A student commented, “How come we can form the quadratic equation in \( m \)? I thought \( m \) is the constant gradient.” In one of the lesson segments, the teacher attempted to direct students’ attention to the varying nature of the tangents’ gradients. “This is \( m_1, m_2 \), this point here got other \( m_1 \), other \( m_2 \). So there are many \( m_1, m_2 \)” The teacher was pointing to different points on the director curve and its corresponding pairs of perpendicular tangents (see Figure 4). By discussing these instances of the director curve problem situation, the teacher was trying to get students to see that the gradients of the perpendicular tangents vary along the path of the director curve. This particular lesson segment drew sudden insight on the part of the students as they gesticulated excitedly “Oh…”

On the other hand, the students’ cognitive dissonance in the formulation of the quadratic equation in \( m \) had also helped to reduce the cognitive demand in following the teacher’s idea that \( m \) is indeed a variable. This could be because the need to resolve their cognitive dissonance allowed them to actively direct their attention to this aspect of the problem situation.
Figure 4. Diagram used in facilitating students’ noticing the gradients of perpendicular tangents $m$ as a variable

Through the enactment of a particular realization of the director curve abstraction as shown in Figure 5, the teacher was also trying to get the students to see that the quadratic equation $(x^2 - a^2)m^2 - 2xym + y^2 - b^2 = 0$ whose coefficients are now fixed, would have two roots in $m_1$ and $m_2$. “At any particular point on the director curve, we would have fixed values for $x$ and $y$ and a pair of $m_1, m_2$ values from the two perpendicular tangents”. Through noticing these instances of the director curve abstraction, the students could reason that the product $m_1$ and $m_2$ from the two perpendicular tangents would be related by the equation $m_1m_2 = 1 = \frac{y^2 - b^2}{x^2 - a^2}$ (see Figure 6).

It then follows that students will notice more effectively if the teacher is able to structure lesson activities in such a way as to bring out the critical aspects of the problem situation through instantiation of its abstraction, as well as their relationships to the solution process. This is especially relevant for students who have yet to develop the executive control (Fan et al., 2002) to resolve competing demands from multiple sources of information in the problem situation. These findings are also consistent with those of Marton et al. (2004) who underscored the importance for
students to notice the relations of parts within the wholes of particular problem situation.

Figure 5. A particular realization of the director curve abstraction

Figure 6. Students’ formulation of the ellipse director curve equation
5 Conclusion

In this study, we presented the need to develop in students the sensitivities to important aspects of the problem situation and noticing such mathematical cues is a critical component of their mathematical reasoning abilities. We had also described three aspects of a mathematics teacher’s classroom practice that can contribute towards facilitating students’ mathematical noticing.

Future research can examine how these findings can be incorporated into the design of professional development programme to develop teachers’ practice for students’ noticing. Another possible future research direction would be to examine the effect of facilitating students’ noticing on their mathematical problem solving competencies, especially on the types of plan they devised and the kinds of problem extensions they generated.

References


Facilitating Students’ Mathematical Noticing


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Chapter 4

Empowering Junior College Students through the Educational Use of Graphics Calculators

Barry KISSANE

Calculators are regularly misunderstood as devices solely for arithmetic computation, including computation in examinations, and even at times regarded as unhelpful for students learning mathematics. Yet, development of graphics calculators in recent decades has been focused on their use as educational devices, and calculator design has been heavily influenced by the needs of senior secondary school students. This analytical chapter describes ways in which graphics calculators can be used educationally to empower students learning mathematics, focusing on a four-part model for calculator use developed by the author and colleagues in recent years. While computation is an important aspect of calculator use, representation, exploration and affirmation are also important, and a graphics calculator will not be likely to empower students learning mathematics if it is restricted to computation.

1 Introduction

In recent years, both in Singapore and in other advanced countries, student experience in mathematics has been affected by the availability of various forms of technology. These have been used both by teachers as part of their instructional process and by students in their role as learners. Arguably the most important of these technologies have been calculators, because they have been sanctioned for use in school curricula, including during high stakes examinations.
Empowering Mathematics Learners

Perhaps inevitably, the use of technology in examinations has given rise to popular beliefs that the main purpose of the technology is associated with examinations, which are generally completed under strict time limits and there is a premium on obtaining answers to set questions efficiently. This chapter has been developed to suggest a different orientation: that the main purpose of the technology is to assist students to learn mathematics well.

The chapter explores the thesis that a graphics calculator can and should be used to empower students to learn mathematics better than would otherwise be possible. A key part of the argument involves a model for the educational use of calculators that, while recognizing that numerical computation is valuable, identifies other aspects of the use of calculators that can be empowering for students. While the focus is loosely on the kinds of mathematics typically encountered at the Junior College level, the source of many of the examples used in the chapter, the argument is intended to apply more generally.

2 Empowerment and Graphics Calculators

The theme of this yearbook is related to the idea of ‘empowerment’. The meanings of words often reveal important attributes, and in this case, there are two distinct meanings of importance. The popular Internet dictionary, dictionary.com, highlights each of two distinct meanings of the verb, ‘empower’:

1. to give power or authority to; authorize, especially by legal or official means:
   *I empowered my agent to make the deal for me.*
   *The local ordinance empowers the board of health to close unsanitary restaurants.*

2. to enable or permit:
   *Wealth empowered him to live a comfortable life.*
   *(www.dictionary.com)*
Each of these two meanings is of importance for the present chapter. The first meaning is reflected in the decision some years ago by the curriculum authorities in Singapore to sanction the use of graphics calculators by Junior College students, through the joint mechanisms of including specific references to their use in the H1, H2 and H3 levels of the GCE A-Level Curriculum and in the associated examinations (Singapore Examinations and Assessment Board, 2015a, 2015b, 2015c). The official documents make it clear that the use of a graphics calculator (without a computer algebra system) is expected of students, and examination papers are set on the assumption that students will have access to a graphics calculator. (e.g., Singapore Examinations and Assessment Board 2015a, p. 3). These decisions effectively empower students to make use of graphics calculators in their studies, and it would be a shallow interpretation of the official intentions to regard the use of calculators to be restricted to examination purposes, as explained by Kissane, Ling & Springer (2015). Without official empowerment of this kind, students would be prevented in practice from use of the technology for learning purposes in classrooms, as can be readily seen by studying countries where that is the case.

While other technologies for mathematics can also be empowering, the significance of decisions to permit the use of graphics calculators is enhanced by the nature of calculators, rather than other kinds of software. As a technology for learning mathematics, graphics calculators enjoy important advantages over computers and tablets: they do not require electricity for operation or frequent recharging, they do not require expensive infrastructure such as wifi or technical support staff, they do not need regular software maintenance or virus protection, and they do not present significant distractions to students through social media or entertainment websites. Unlike more powerful alternatives, graphics calculators have been specifically designed for use in classrooms, especially mathematics classrooms, and so incorporate features of direct significance to the school mathematics curriculum.

The second dictionary meaning of empowerment is of most importance for the present chapter, however, as it focuses on what the students themselves are empowered to do as a consequence of the official framework. As the syllabus documents make clear, the use of graphics
calculators is intended to be of educational significance, and not just of significance for examinations. Indeed, for many naïve users of calculators, the technology might be regarded as principally concerned with (numerical) ‘calculation’, rather than recognizing a wider educational significance. Hence a key purpose of this chapter is to highlight some of the ways in which students can be empowered to learn mathematics differently when access to a graphics calculator is assured. Towards this end, a model for the educational use of calculators is briefly described in the next section of this chapter, and then elaborated and exemplified in later sections.

3 A Model for the Educational Use of Calculators

When calculators are permitted for educational use, it seems that there is a degree of unease about their use, and an implicit assumption that the major function is to ‘calculate’. (This assumption would be unsurprising in view of the use of the label ‘calculator’, of course.) The calculator is often described in terms of prohibitions and limitations, as for the Guidelines published by the Singapore Examinations and Assessment Board (2016), which have the particular purpose of ensuring that examination use is properly prescribed. While early calculators (around forty years ago) may have been restricted to supporting numerical calculation, development of calculators since that time, and the work of teachers to incorporate them into educational curricula, has allowed a much more expansive view of their role in education to be adopted.

Partly in response to the limited thinking that calculators are solely instruments for numerical calculation, Kissane & Kemp (2014a) proposed a four-part model for the educational use of calculators. This model will be adopted as the framework for the present chapter, concerned with the use of graphics calculators with senior students. While details are available online, the following brief overview of the four parts of the model will suffice to orient the reader.

*Representation:* Calculators provide a means for mathematical concepts to be represented on a screen. While modern calculators increasingly represent mathematical notation in conventional ways, the
significance of representation goes beyond the surface level of physical appearance. Thus, for example, mathematical functions can be represented in a variety of ways, helping students to develop an understanding of the meaning of a function. Similarly, a sophisticated concept such as a differential equation can be represented as a slope-field diagram, providing students with a fresh way of thinking about the concept.

Computation: While it is very limiting to regard calculators as solely valuable for numerical calculation, it is nonetheless important to note that they are capable of performing calculations, and that such a capability is of assistance to learners. One aspect of the significance of computation is that practical problems can be resolved and thus applications of mathematics and mathematical modelling can be routinely included in the curriculum, when a reliable and efficient means of calculation is available. Without access to adequate computational capabilities, effective work in statistics with real data is essentially prevented. In addition, the availability of a means of computation brings fresh arguments regarding which aspects of mathematics are best handled by hand and which routine computations can be better left to a machine, so that the importance of analysis and mathematical thinking are adequately recognized.

Exploration: Calculators empower students to engage in various forms of mathematical experimentation, exploring ideas and relationships for themselves. This requires a reorientation to a calculator to regard it as a device to support learning, rather than mainly as a device to produce a numerical result, after all the necessary mathematical work has taken place. Modern calculators have been designed to accommodate such activity, which generally relies on teachers helping students to make use of the technology in this way. Some exploratory work relies on the capability of calculators to represent concepts efficiently and in various ways, while other work depends on the availability of structures such as spreadsheets, programming languages or random number generation.

Affirmation: The fourth aspect of the model concerns the ways in which students can use a calculator to affirm – or, indeed, to contradict – their mathematical thinking. At a trivial level, a calculator can be used to ‘check’ a numerical answer to a mathematical question, which provides a kind of reassurance for a student. However, more powerfully, students can use a calculator to engage in hypothetical or conjectural thinking, which
often involves active thinking to use the calculator in fresh ways. For example, an important activity for senior students involves examining or even deriving identities and seeking a formal proof for their veracity. A separate, but related activity might involve students using graphs or tables to understand the meaning of equivalence and to have their formal work affirmed and made practical. Again, and as for exploratory work, such activity is best stimulated by teachers supporting students to regard their calculators as tools for thinking and predicting, not only as tools for calculation.

In the following section of this chapter, these four model components will be exemplified and explored in further detail, using examples drawn mostly from senior Secondary or Junior College mathematics curricula.

4 Examples of Empowerment

In this section, examples of ways in which a graphics calculator can be used to empower learners will be described. An over-riding perspective of these examples is that the primary role of the calculator is to help create meanings for mathematical objects, rather than merely to provide numerical answers to questions of interest.

For convenience, and to be sympathetic with Singapore requirements at present, a CASIO fx-9860GII series calculator is used to provide the examples offered, but other recent models and recent models from other manufacturers might have been used; the illustrations provided do not depend critically on particular choices. Indeed, current calculator models from various manufacturers (not presently approved for use in Singapore) could be used to give even more examples than those offered in this chapter.

4.1 Representation

Modern graphics calculators have been developed to use conventional mathematical symbols in order to represent ideas. Developments of this kind are relatively recent, but have the effect of aligning what students see on their calculator screens with what they see in their textbooks and on
classroom whiteboards. Two examples of this are offered in Figure 1, the first showing how powers and radicals are connected while the second screen shows a conventional way of representing an interesting series. In these two cases, the first line of the screen is entered by the student, while the second line is the response provided by the calculator; of course, students need to learn how to enter relatively complicated expressions on their calculator, which is greatly facilitated by the match between conventional and calculator notation.

In an analogous way, Figure 2 shows how a matrix and its inverse are represented, illustrating the use of matrix terminology as well as the use of fractions in the inverse matrix. In this case, the calculator shows the conventional use of an exponent of -1 to represent an inverse, but uses a different way of referring to a matrix from the conventional methods (such as use of bold type for the matrix name). Such minor variations are increasingly rare on modern calculators and students generally have little difficulty in adapting to them.

While it is helpful, and even reassuring, for a calculator to represent mathematical symbolism conventionally, there is a more important meaning of ‘representation’ that offers empowerment of learners. One way
of characterizing this difference is to regard the word as ‘re-presentation’, or presenting again, reflecting the calculator’s preference to represent a mathematical expression in a different, but equivalent, form from the original form. Figure 3 shows two examples of this kind of phenomenon.

In the first screen in Figure 3, the calculator has re-presented \(\sqrt{48}\) as \(4\sqrt{3}\), which might reveal to a student that there are two different representations of this number. Also shown in this screen is another possible re-presentation, showing the result as an approximate decimal number rather than a surd; this representation was not automatic, but provided by selecting a decimals to fractions conversion key, empowering students to choose for themselves which kind of representation of a number suits their purposes.

The second screen in Figure 3 shows an example of a more sophisticated re-presentation, in which the calculator has shown a complex fraction in the form of a number with separate real and imaginary parts. Once again, the two representations are merely equivalent forms of the same number. The further entry in Figure 3 is included to show how the calculator routinely re-presents a relevant binomial complex product, providing the potential for some insight into the calculator’s means of representing a number so that it has no complex component in its denominator.

While these examples of multiple representations of expressions are relevant to students studying at the Junior College level (and beyond), they illustrate capabilities of modern scientific calculators, not only the more powerful graphics calculators that are the main subject of this chapter. In doing so, they illustrate the fact that a modern graphics calculator includes
Empowering JC Students through Graphics Calculators

To illustrate the most important of these opportunities, consider the three ways in which a function can be represented: symbolically, numerically and graphically. A key empowering capability of a graphics calculator is that all three of these representations are readily available to students, and they are able to move freely between the representations. Figure 4 shows the first two representations for the case of a linear function, \( f(x) = 2x - 1 \).

![Figure 4. Representing a linear function symbolically and numerically](image)

The first (symbolic) representation is slightly different from the intended \( f(x) = 2x - 1 \), but this ought not be regarded as problematic: in fact, it helps students to appreciate that the same function can be represented symbolically in more than one way. The second (numerical) representation empowers students to see that a function can be regarded as a set of ordered pairs, some of which are shown on the screen. This representation also serves the useful purpose in this case of making visible that the values of the function change in a regular way, increasing by 2 when the \( x \)-value increases by 1, a characteristic property of linear functions, but not of other functions.

The third (graphical) representation of this function is shown separately in Figure 5, with two slightly different screens shown. Each of the two screens shows the characteristic linear shape of this function. The two screens also illustrate that the representation of the function can be changed slightly by moving the cursor to a new position, to examine the graph in more detail. The second graph is shown with a coordinate grid to illustrate a cosmetic alternative for this representation. Of particular importance is the observation that the graphical representation also shows
the other two representations simultaneously, as the ordered pairs are shown in the form of coordinates at the bottoms of the screens while the symbolic representation is visible at the tops of the screens.

Multiple representations of functions empower students to think about them in ways that were difficult to achieve before the availability of technology of these kinds, allowing free movement between representations and permitting students to manipulate the representations in order to understand the relationships between them.

Space precludes a complete treatment of the ways in which graphics calculators can represent mathematical concepts. However, two more examples related to calculus are presented briefly in Figures 6 and 7. In the first screen in Figure 6, a representation is given of a definite integral as an area under a curve and above the x-axis; the second screen shows how a graphics calculator can represent the same integral symbolically and numerically (and with a slightly different – exact – result).

The screens in Figure 7 suggest how representations of the important exponential series can help students to appreciate the critical concepts of a limit and of convergence. The calculator has been used to tabulate successive terms of the sequence of inverse factorials, as well as the
associated series. The first screen shows that successive terms decrease in size, as the value of \( n! \) increases very rapidly. A representation such as this empowers students to see how each term of the series is the sum of the previous series term with the addition of the next term of the sequence. In the second screen, the representation has been changed by scrolling down a little, and reflects the remarkable fact that the successive terms of the series are unchanged after the twelfth term, at least to the degree of precision shown at the bottom of the screen. These representations provide a tangible sense of the key concept of convergence to a limit as well as the limit itself, in this case Euler’s remarkable constant \( e \).

**Figure 7.** Representing convergence to a limit of the exponential series

In summary, modern graphics calculators offer students important and conventional representations of numbers and mathematical concepts, giving rise to the first component of the model for the educational use of graphics calculators. Although they are sometimes referred to as ‘graphing’ calculators because of their ability to draw graphs of functions, it is important to recognize that the graphics screen on graphics calculators offers many ways of empowering students through representations, and it is unwise to focus on only one manifestation of those capabilities.

### 4.2 Computation

Although it is suggested that the most valuable purpose of a calculator is not numerical computation, it should be recognized that access to computation can be empowering for some purposes. Many calculations that students encounter can and should be done mentally, and many others can sensibly be completed with good approximations; in some cases, written methods of computation may even be appropriate.
Notwithstanding all of these, there will still be situations where the calculator will be empowering because of its capacity to undertake a calculation.

Indeed, practical aspects of mathematics that involve real measurement or real data, as distinct from artificial examples that do not involve the real world, will require access to calculators. Similarly, trigonometric computations will not routinely involve the few angles for which exact ratios are known. Without access to some technology for calculation, students will not be able to undertake mathematical modelling using real data that they have obtained themselves or via third party sources, such as the Internet or authoritative publications.

As an illustration, consider the task shown in Figure 8, reliant on data from the Statistics Indonesia census.

According to the Statistics Indonesia census in 2010, as reported by *Wikipedia* (2016), Indonesia’s population was 237 641 334, and the population growth rate was 1.54% over the years 2000-2010. What should the population be expected to be in 2016?

*Figure 8. Indonesian population task*

A modelling task of this kind ought not be completed by hand, but a calculator offers students a variety of ways of completing the calculations. A key aspect is an assumption that the population growth rate over the years from 2000-2010 will also apply for the following years 2010-2016, so that a population prediction can be made. There are several ways in which such a prediction can be made, the least sophisticated of which involves (tediously) repeated addition of an annual increase of 1.54%. Among more sophisticated approaches are the use of a single exponential calculation, as shown in the first screen of Figure 9 or the tabulation of the corresponding exponential function, as shown in the second screen in Figure 9. The latter approach allows the increases in population to be more easily seen.
Notice that the use of a calculator here demands that students consider efficient representations and reasonable assumptions, so that numerical responsibility has not been entirely ceded to a machine. In this case, the 2010 population has been stored into a calculator variable \((A)\) for ease of operation. Students will need to consider the results carefully, too: the population of Indonesia is of necessity an integer, while the calculator results are non-integral. Basic numeracy is required also to recognize that an increase of 1.54\% can be represented by multiplying by 1.0154, instead of calculating an increase and then adding it to the existing population. In addition, work of this kind can be used to encourage students to think about the reasonableness of any results; the expected population of 260 462 339 might sensibly be rounded in some ways as acknowledgement of the unavoidable errors involved in such work. Of course, alternative assumptions about the growth rate in the years 2010-2016 can be made, and easily accommodated on the calculator with an appropriate change of parameters.

Other kinds of tasks associated with the same context also require access to appropriate calculations, which can be handled well with a graphics calculator. Figure 10 shows some appropriate questions that in this context students might reasonably be expected to address.

When is the population of Indonesia likely to reach 300 million? What will be the effect of a reduction in the annual population growth rate to 1.2\%
A characteristic of graphics calculators is that students are frequently able to address a task in several different ways. Thus, (naively) assuming that the population growth rate remains constant at 1.54% per annum, students might construct a suitable table, similar to that shown in Figure 8, and inspect it to determine when the population is close to 300 million. This approach is shown in the first screen in Figure 11. But there are also other alternatives, such as using a graph of the projected population after 2010, using an automatic equation solver, or even considering an exponential expression using logarithms to base 1.0154, as shown in the second screen in Figure 11. In either case, the resulting projection is that the population will reach 300 million a little over 15 years from 2010, or around 2025. Clearly, work of this kind is not sensibly conducted without the use of technology, and it seems reasonable to conclude that the calculator empowers students to both answer the question asked and consider alternative ways of mathematical thinking about it.

Statistical work is a special case of these observations. If students are to study statistics, it is important that they have the experience of gathering and analysing their own data or dealing with real data collected from a credible source. To continue the Indonesian example, students might choose to consider the population growth over time, with a view to predicting future changes, as an alternative to modelling with population growth rates. Table 1 shows a summary of Indonesian population in the census years since 1971, according to Statistics Indonesia, which can be used for this purpose.
Table 1

*Historical population of Indonesia*

<table>
<thead>
<tr>
<th>Year</th>
<th>Population</th>
<th>% p.a.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1971</td>
<td>119,208,229</td>
<td></td>
</tr>
<tr>
<td>1980</td>
<td>147,490,298</td>
<td>+2.39%</td>
</tr>
<tr>
<td>1990</td>
<td>179,378,948</td>
<td>+1.98%</td>
</tr>
<tr>
<td>2000</td>
<td>206,264,595</td>
<td>+1.41%</td>
</tr>
<tr>
<td>2010</td>
<td>237,641,326</td>
<td>+1.43%</td>
</tr>
</tbody>
</table>

Source: Statistics Indonesia

A graphics calculator can be used by students to find a model for the population data and to then use the model to make suitable predictions. Tedious computations with real data can be left to the calculator, so that students can focus their thinking about the issues involved with predictions of the kinds contemplated. In this case, the data suggest that a linear relationship adequately describes population growth over the first forty years, as suggested by the scatterplot in Figure 12.

*Figure 12. Linear population growth for Indonesia 1971-2010?*

The resulting prediction for the population in 2016 is about 256 million, shown in Figure 13; this is close to the prediction shown in Figure 9. However, Figure 13 also shows that the same model predicts that the Indonesian population will reach 300 million in 2031, some six years later than the predictions shown in Figure 11; this is a substantial and disturbing discrepancy.
A graphics calculator empowers students to undertake calculations of these kinds, not realistically possible without the support of the technology. In this case, a more realistic model for the population growth over time, and more defensible for the always risky strategy of extrapolation, is an exponential model. Again, the necessary calculations are best handled by the technology, and the results shown in Figure 14 suggest that the population might reach 300 million as early as 2022.

In this case, the data shown in Table 1 suggest that, because population growth rates are clearly not constant over time, neither a linear nor an exponential model will be entirely satisfactory. The role of the calculator here is to offer students sufficient numerical support to focus their thinking on such issues, rather than being preoccupied with the computations.

The availability of a computational device like a graphics calculator has a further kind of implication for Junior College and similar courses. When significant calculations can be automated, such as the numerical solution of equations or integration by numerical methods, students might be empowered to undertake mathematical work not otherwise available to them, when restricted to analytic approaches. This issue is explored briefly in Kissane and Kemp (2014a).
In summary, this example illustrates that a graphics calculator empowers students to undertake calculations not easily accessible to them otherwise, and empowers them to deal with mathematical tasks in more than one way, both powerful for learning.

4.3 Exploration

Arguably the most important way in which a graphics calculator might empower students is through providing mechanisms for the exploration of mathematical ideas. The essence of this idea is that, with some direction, students can use a graphics calculator to explore aspects of mathematics by themselves, and learn from the resulting manipulations. The calculator is a device for learning about mathematics and mathematical relationships, not merely a device for obtaining answers to questions. An extensive collection of examples of this kind of exploration is given in Kissane and Kemp (2014b), at the conclusion of each chapter focusing on particular mathematical topics. In this section, space is available for only a few examples.

Perhaps the most obvious avenues for exploration are associated with the study of functions and their graphs. Because a graphics calculator can quickly and efficiently be used to generate a graph of a function, students are empowered to explore graphs and tables of values by themselves to learn about the associated mathematics. This includes shapes of families of functions, relationships between roots of functions and the associated equations and the relationships between functions and their extrema.

As Figure 15 suggests, some graphics calculators permit animation of graphs, an alternative to exploring a succession of individual graphs to see the effects of parameters on graphs and tables.

![Figure 15. Exploring graphs of parabolic functions with animations](image-url)
An especially productive form of exploration involves the study of transformations of functions, routinely included in curricula at the senior secondary school, including the Junior College curricula, to examine relationships between the symbolic form of functions and their properties. This idea can be used for other purposes, however, such as understanding the nature of derivative functions. Figure 16 shows in the first screen a cubic function and a transformation to generate its (numerical) derivative function; the associated graphs of these are shown in the second screen.

![Figure 16. Using a transformation to explore a derivative function](image)

From such work, students can see that the turning points of the cubic are associated with the zeroes of the derivative function (shown as a bold graph in Figure 16), that the sign of the derivative function reveals the rate of change of the function itself and explore the meaning of a point of inflection. Changing the first function in the list empowers students to explore readily in this way a range of cubic functions – or indeed any other functions.

In a similar way, ideas that are frequently regarded by students as essentially symbolic in nature can become more tangible, with calculator explorations. A good example involves the idea of differentiation from first principles. Figure 17 shows how a transformation can be used to explore this (for the sine function). The bold graph shows the rate of change using first principles, with \( H = 0.1 \). In this way, students can see for themselves that the derivative function is close to the familiar cosine function. Of course, the limiting case, as \( H \) tends to 0, can only be approximated, but students can explore this for themselves by changing the value of \( H \).
Apart from visual checks of this kind, the same idea can allow students to explore the difficult concept of ‘closer and closer’, associated with limits and differentiation from first principles. To suggest the direction for such explorations, Figure 18 shows tabulated values of the sine function as $Y_1$, the rate of change function as $Y_2$ and also the expected limit, the cosine function, as $Y_3$. In the first screen, $H = 0.1$, while in the second screen, $H = 0.0001$. By exploring such a context, students are empowered to see that, even when the tabulated values of $Y_2$ and $Y_3$ look to be the same, they are not quite the same, even as $H$ decreases and gets closer to the limiting value of zero.

Empowerment can arise from having a variety of ways of addressing mathematical issues, instead of just the ‘one, right way’ that students often associate with mathematics. A graphics calculator often provides students with a platform for explorations of many kinds, such as with a task of “Solve the equation $x^2 + 2^x = 2$, in as many ways as you can.” Figure 19 shows two of the many alternative approaches, the first using an automatic equation solver, and the second using a pair of graphs. In this case, the second approach makes it clear that there are at least two solutions to the
equation. The graphics calculator empowers students to exercise their imaginations and to explore the situation from several perspectives.

![Figure 19. Exploring some solutions to \( x^2 + 2x = 2 \)](image)

As well as features built into the devices by manufacturers, graphics calculators generally allow users to develop new features through the use of short programs, written by users or obtained from others (such as teachers). One example of this involves the idea of a direction field diagram, useful for visualizing a differential equation and generating numerical solutions to the equation. An elementary example is shown in Figure 20 for the differential equation, \( y' = x \). The first screen shows a direction field diagram, with the derivative at various points on the plane shown with a short line segment. The second screen shows approximate graphical solutions to the equation for two separate boundary conditions, \( y(-2) = 3 \) and \( y(-2) = 0 \). In each case, the characteristic parabolic shape is suggested by both the diagram and the solution curves.

![Figure 20. Using a direction field diagram to explore the differential equation \( y' = x \)](image)

Explorations are not restricted to the study of functions and calculus of course. To illustrate a different field of mathematics, Figure 21 shows some results of using a small program to study the key statistical idea of a sampling distribution. The program takes a number of samples at random from a given finite population and studies the distribution of the means of
the samples. The first screen shows the original population, the most recent (last) sample and the sample means. (Only the first few lines of each set is shown.). In this case, fifty separate samples were taken at random from the population in the first column, with the sample means shown in the third column.

Figure 21. Exploring sampling distributions via random samples

The idea of random sampling is absolutely essential for the understanding of elementary statistical inference (which relies on theoretical results for an infinite number of samples). The graphics calculator in this case empowers students to appreciate the underlying results associated with the Central Limit Theorem, and to see for themselves that the actual results of taking random samples varies each time (and also varies between fellow pupils). Yet, there is a recognizable consistency in the results obtained, and the beginnings of a normal shape, even with as few as fifty samples, as in this case. Students can use this environment to explore the effects of changing the sample size and can compare parameters of the (finite) sampling distribution with the theoretical results to add meaning to the ideas involved.

In summary, these few examples have illustrated that the graphics calculator can be used productively by students to explore important mathematical ideas for themselves, once they have mastered the operation of the device. In essence, the graphics calculator empowers them to engage in fruitful explorations, with suitable guidance from their teacher.

4.4 Affirmation

The final aspect of the proposed model involves students using the calculator to affirm or to contradict their thinking. Again, the central idea
here is that the calculator is part of the environment in which students are learning mathematics, and is not restricted to its role in examinations or in merely evaluating a numerical answer.

The important idea here is that students need to think about what they are doing and not merely use a device thoughtlessly. Tasks can be designed for this purpose, and students encouraged to clarify what they expect to happen, to allow their thinking to be affirmed or to be contradicted. Figure 22 shows two examples of this aspect. In the first screen, before evaluating the square root of 8, students should ask themselves what result they expect the calculator to produce; many will be surprised that it is not a number, but an expression, and should be encouraged to think about why the result shown is obtained.

The second screen in Figure 22 shows graphs of a pair of functions, \( f(x) = x(x + 1) \) and \( g(x) = x^2 + x \). Some students will be unsurprised by what they see, while others will be looking for a second graph, expecting that these two functions are different in some important way.

Many, if not most, of the examples of calculator use described earlier in this chapter can be regarded in this sort of way, provided students are encouraged to be active and thoughtful users of the devices, which usually requires teachers to help them attain such an orientation. While sophisticated calculator users, including teachers, are inclined to anticipate what will happen on their calculators, many students do not naturally do so. For this reason, encouragement in the form of questions such as those in Figure 23 might be important in the classroom to develop this orientation and to empower students to learn from the results of their actions with the calculator.
What do you *expect* to happen? (Before you enter a command)
What *actually* happened?
Why was that particular result obtained? What does it mean?
Were your expectations affirmed? Contradicted?
Were you surprised at the result? Why?
How could you make a particular result …?
What will happen if you …?

*Figure 23. Examples of internalised questions for using a calculator thoughtfully*

In summary, this final aspect of the model is concerned with the orientation of the user of graphics calculators. A disposition for thoughtful use of calculators can empower students to learn many aspects of mathematics.

### 5 Conclusion

In this chapter, the educational use of graphics calculators has been explored, drawing on a four-part model, to demonstrate that learning of mathematics can be empowered by access to such devices. The examples used are generally concerned with a senior mathematics curriculum, such as those for the Junior College, although the principles suggested can be applied across a range of courses.

### References

Kissane, B., & Kemp, M. (2014a) A model for the educational role of calculators. In W.-C. Yang, M. Majewski, T. de Alwis & W. Wahyudi (Eds.) *Innovation and Technology*


Chapter 5

Understanding Future Teachers’ Mathematical Knowing to Overcome Double Discontinuities

Hyungmi CHO     Oh Nam KWON

This study explores the effectiveness of university content learnt in Korean teacher education institutions in order to better understand the chronic problem of the discontinuity between university and school mathematics. The study selected school mathematics content for which university mathematics was applicable. Questions that could be solved using concepts from university mathematics were developed, following which the novice teachers were asked for reasons guiding their answers. The results suggest that in order to adapt university mathematics to teach school mathematics, the following three understandings are needed; a rigorous understanding of mathematical theorems, an understanding of extension of mathematical definitions, and a rigorous understanding of mathematical definitions. The results of the study suggest that it is not necessary to increase the content of university mathematics such as algebra and analysis. Instead courses necessary for future teachers to understand such university mathematics should be increased. The study hopes to be a reference for future teacher educators constructing content for their mathematics courses.

1 Introduction

Empowering students’ mathematical thinking cannot be perceived independently of the problems in mathematics teachers’ competency.
Teacher education institutions in particular play a key role in fostering teachers’ professionalism and competence. The mathematics teacher education curriculum can be largely divided into university mathematics content knowledge (CK), pedagogical content knowledge (PCK), practicum to gain practical knowledge via field work, and general pedagogical knowledge (GPK) (Kwon & Ju, 2012). According to Kwon, Kim and Cho (2012), who analysed the education curriculum of colleges of education in Korea based on the above frame, courses relating to CK and PCK have a ratio of 5:1, and 83% of all the lessons are instructor-centred. The study revealed the need for a greater number of PCK lectures related to practical school teaching and the need to further investigate the content necessary for high quality education. The need to conduct vigorous research on the content and methods of PCK courses has also been highlighted, along with the need for content and pedagogical methods helpful in relating to school mathematics in CK courses.

Several studies have analysed the educational curriculum and content to improve the quality of mathematics teachers’ education provided in teacher education institutions. Although some changes have been made, the effectiveness of teacher education institutions in fostering the professionalism of teachers remains doubtful.

Felix Klein, a German mathematician in the early 20th century, explained the gap between future teachers experiencing advanced mathematics at university and school mathematics handled when they go back to school as a “double discontinuity” (Klein, 1924). The first discontinuity refers to the difficulties in terms of the precise and formal mathematical practice experienced by future teachers when they become university students. The second discontinuity is experienced by teachers when they return to schools and teach in the ways they were taught in high school and when they find the advanced mathematical experiences learnt in university irrelevant to teaching.

School mathematics appears as a result of instructional transformation by teachers. Teachers need to be able to teach mathematical content and make it accessible to their students in a didactically well-prepared manner. The second discontinuity problem emerges because teachers’ knowledge learned in university is not included in their practice.
Future teachers are expected to be able to apply the advanced mathematical experiences meaningfully in practice having learnt the educational courses of CK and PCK in the university. However, teachers’ knowledge is not naturally transformed to fit school mathematics nor reconstructed into the learners’ perspectives simply by learning CK and PCK. This is indicated by the fact that double discontinuity continues to be a topic of great concern.

This study explores teachers’ knowledge in an attempt to find a way to empower future teachers who are learning how to teach mathematics. Mathematical empowerment refers to enhancement of attributes that includes mathematical language, skill, using mathematics and applications, and the process of gaining and imparting power to future teachers individually or as a group within educational institutions, or the aiding process for them to gain the power (Ernest, 2002).

Recently, as an extension of Mathematical Knowledge for Teaching (MKT) study, Heinze, Lindmeier and Dreher (2015) conceptualized School-Related Content Knowledge (SRCK) as “knowledge to correctly apply academic mathematical knowledge in the school context.” This was considered a special mathematical content knowledge distinct from the previously studied CK and PCK, and they posited this knowledge as the teacher knowledge required to bridge the gap between university mathematics and school mathematics.

This study explores the educational content involved in empowering the relevant knowledge, but is not aimed at expanding the subject-matter knowledge to be studied at university mathematics such as algebra or analysis. Rather, it aims to emphasize the behavioural aspects that can help in the learning of university mathematics processes by future teachers, and focuses on knowing the processes to acquire rather than merely declare SRCK.

2 Research on Mathematics Teacher Knowledge

Many attempts have been made to conceptualize mathematics teachers’ professional competency based on empirical and experimental cases. Some studies such as Mathematics Teaching in the 21st century (MT21)
Empowering Mathematics Learners (Schmidt et al., 2007) and the Teacher Education and Development Study in Mathematics (TEDS-M) (Tatto et al., 2012) carried out international comparative studies on teachers’ competency by developing conceptual frameworks for teachers’ competency and assessment questions based on this framework (Blömeke et al., 2008, Blömeke & Delaney, 2012).

In TEDS-M, competence is defined as those latent dispositions that enable professionals to master their job-related tasks. Teacher knowledge as one facet of competence was conceptualized in TEDS-M. TEDS-M divided teacher knowledge into MCK and PCK, and each of these into three subdivisions, following which tasks were developed and assessment was carried out.

MT21 examines how teacher preparation is done differently across several high-achieving countries. In this study, teachers’ knowledge was largely divided into CK and PCK, where PCK includes instructional planning, student learning, and curricular knowledge as its subcategories. Professional competence in MT21 and TEDS-M includes subject-related and interdisciplinary cognitive dispositions of performance, as well as affective-motivational beliefs as part of a teacher’s personality. Research on teachers’ knowledge started out as Pedagogical Content Knowledge conceptualized by Shulman (1986, 1987). Shulman (1986) explained PCK as a special form combining the concepts of content knowledge and pedagogical knowledge. Since then many researchers have attempted to conceptualize PCK for various subjects based on the definition of Shulman.

The most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others (Shulman, 1986, p.9)

Emphasizing the role of teachers, mathematics education researchers have studied the knowledge required for teaching mathematics. Ball and her colleagues (2005, 2008, 2009) developed MKT, a theoretical framework for professional content knowledge for mathematics teachers’ effective teaching, and took empirical measurements. Knowledge types necessary for teachers to teach mathematics at primary schools in the
United States were investigated in order to clarify the components of MKT. The teacher knowledge was conceptualized into six areas, where subject matter knowledge and pedagogical content knowledge are the upper domain, each consisting of three subdomains.

However some studies have pointed out that the PCK learned by teachers in teacher education institutions, often fails to integrate with their teaching practice, because elements of PCK are assimilated structures of procedural mathematical understanding that lack consistency (Putnam, Heaton, Prawat, & Remillard, 1992). Thompson (2015) argued that knowing-to and knowing-why are the most important forms of knowing for teachers and that focusing on teachers’ mathematical meanings is useful for understanding their instructional decisions, both in planning and in moments of teaching. In terms of radical constructivism, he asserted that presented teacher’s knowledge is transformed according to contexts and that it is difficult to explain teachers’ behavior and decision making with MKT research as declarative knowledge. He adduced that PCK must be interpreted as the fundamental transformation of existing knowledge or as a result of the formation of new knowledge. At this moment, knowledge cannot be separated from the orientation and goals of the subject, and he takes the radical constructivist stance of knowledge as justified true belief of the cognitive subject. Therefore, the meaning of the mathematical concept of the individual teacher becomes important to describe the actual teaching.

Teachers’ mathematical meanings guide their instructional decisions and actions (Thompson, 2013). His research group developed the Mathematical Meaning for Teaching Secondary Mathematics (MMT-SM), a diagnostic instrument designed to provide insights into the mathematical meanings with which teachers operate. MMT-SM better explains teachers’ decisions and the process of instructional transformation that occurs in the classroom than does the MKT. However, MMT-SM does not provide a method to solve the gap between university mathematics experience and teaching school mathematics for future teacher education. Recently, as an extension of MKT, Heinze et al. (2015) conceptualized SRCK as “knowledge to correctly apply academic mathematical knowledge in the school context.” This was considered a special mathematical content
knowledge separate from the previously studied CK and PCK. Whether it is a new domain of knowledge that has not yet been researched or whether it can be separated from PCK conceptually is controversial.

However, within the knowledge of the domain to be discussed, i.e. in order to overcome the gap between university mathematics and school mathematics, in that it newly evokes a discussion about the knowledge of the teacher, it is worthwhile to explore SRCK. The SRCK lens in this study is useful to view both university mathematics and school mathematics at the same time. This study explores a method to understand the components of SRCK that can be used in future teacher education institutions.

3 How to Connect Between University Mathematics and School Mathematics

In this chapter, we suggest three ways of understanding university mathematics in order to be able to use it while teaching school-math. The research team has selected tasks that can reveal SRCK by reviewing the tasks that have so-far been developed in order to assess teachers’ knowledge, and revised them to suit our purpose. Additionally, new tasks were developed. The mathematical content used in constructing the revised tasks and selected tasks pertain to middle school mathematics where mathematical concepts and definitions from university mathematics can be used.

Eight secondary school novice teachers who had taught for less than a year after graduating from teacher education institutions were asked to complete the tasks. Novice teachers were chosen since they seemed the most suitable in terms of recency of exposure to university mathematics and actual teaching of school mathematics. Thus deficient areas of knowledge can be identified and recommended for inclusion into appropriate courses of study at teacher education institutions. In Korea, preservice teachers must complete the required CK and PCK courses to earn 50 credits in total in order to get a teacher certification. They must complete more than 7 subjects (more than 21 credits) among the 11 fundamental subjects; methodology, number theory, complex analysis, analysis, linear algebra, abstract algebra, differential geometry, geometry,
topology, probability and statistics, and graph theory. Also, they are required to complete at least 8 credits of PCK courses. The 8 teachers in this study qualify for teacher certification. So we assumed that there is not much difference in the quantitative level on the advanced mathematics experience. They were first asked to solve the tasks, following which semi-structured interviews were conducted to learn about the reasons for their answers and possible didactical methods assuming they would teach students the mathematical concepts and theorems involved in tasks they solved. The teacher participants are given the pseudo-names Teacher 1, Teacher 2, …, and Teacher 8. All responses are transcribed and categorized.

3.1 A rigorous understanding of mathematical theorem

Limit is a typical concept that is handled in both school mathematics and university mathematics; however, it is handled in significantly different ways in school and university in terms of its respective content and approach. The concept is defined in a concrete and intuitive way in school mathematics, while it is defined in an abstract and formal way in university mathematics.

Figure 1 is a task for school mathematics based on the concept of limits and quadratic equations.

![Figure 1. Task for a rigorous understanding of mathematical theorems](image)
The existence of a solution must be checked in order to find the value of the given infinite sequence \( \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}. \) That the infinite sequence is upper bounded and monotonically increasing must first be shown. Then, either using the monotone convergence theorem or completeness axiom the upper bounded and monotonically increasing infinite sequence is then shown to converge. The solution given in the task made an error in that it assumed the convergence of the infinite sequence and found the solution of the quadratic equation by applying the properties of a converging sequence without first proving the convergence. In determining the error of this solution, the teachers responded in one of the following ways.

- It’s perfect. No error.
- If it does not converge, the value of this infinite radical cannot be defined. Therefore, assuming that the infinite radical converges is wrong.

A few of the teachers who were unable to find the error explained that they assumed that “find the value” problems “already have” a solution. Therefore, there was no need to determine the convergence of the given value, and they solved it assuming the value exists. They thought there was no error in the given solution. These decisions imply that the teachers had used the familiar “school mathematics.” School mathematics usually ask to “find the value” in problems finding the convergent value, while university mathematics problems state “determine whether the given sequence converges or diverges, and find the value if it converges” in problems to find the limit of an infinite sequence. Since the convergence would be assumed for finding the limit problems in school mathematics, the teachers made such decisions. In other words, the teacher already had a specific understanding of the limit problem. This can be considered as a certain orientation of “finding the value assuming that it exists in a problem for finding the limit.” However, for others who answered that there was no error, the question became a simple one that asked to express the wanted value with symbol “\( x \)” and solving of a quadratic equation.
Hence, the meaning of convergence and the concept of infinite sequence were not considered while solving the question. The teachers who answered that the convergence needed to be checked had not only found the error in the solution but also mentioned the concepts needed to find the value from using university mathematics. One teacher who pointed out the error answered as follows.

Teacher 3: This is ‘smaller than 2’. The number is getting bigger. An increasing sequence that has an upper bound is convergent. So the infinite radical is convergent.

Although it did not provide the perfect calculation required to determine whether the infinite sequence is smaller than 2 or increasing, the exact concept needed to determine the convergence was explained. This teacher not only understood the application of a mathematical theorem, but also understood its conditions. Depending on how precisely one has understood a mathematical theorems, the given task can simply be a problem to solve a simple quadratic equation or a problem for applying the properties of a converging sequence.

### 3.2 An understanding of extension of mathematical definitions

Mathematical definitions are not arbitrarily given and have their own context and meaning. However, it has been pointed out that students consider mathematics as eternal truths and given objects while they learn school mathematics.

Task 2, shown in Figure 2, is about an operation related to 0. The task aims to check whether the teacher would define 0^0 and what the definition would be if so and what the reason would be if not. The 0^0 task is extracted from research by Borasi (1996) on the difficulty faced by high-school and university students in handling errors. The task related to the greatest common divisor was developed by the research team as an extension of Borasi’s idea.
How do you explain the following questions to the students: “What is \(0^0\)?” and “What is the greatest common divisor of 0 and 0?”

*Figure 2. Task for an understanding of extension of mathematical definitions*

The responses to the first question were classified into the following four types.

- I don’t know.
- 0
- 1
- It cannot be defined

The teachers who responded, “Zero to zero power? It’s not number,” “It’s impossible to calculate,” and “The base cannot be 0” during the interview were classified as “I don’t know.” They knew that 0 could not be a base but they couldn’t adduce any reason for that. This means that although they knew the mathematical definition, they couldn’t explain how to extend the definition.

The teachers whose responses were classified as ‘0’ are those who considered both 0’s as nothing and then defined \(0^0\) as 0. Since this question is not a result of abstraction on a concrete objects, analyzing 0 as a quantity is inadequate. Some of them responded \(0^0 = 0\) because they remembered the law \(a^0 = 1\) where \(a\) is a non-zero real number”. Some of the teachers who answered “1” extended the exponent \(a^0 = 1\) where \(a\) is a non-zero real number” to the case of 0. Some of the teachers who answered “1” explained by saying “I remember that any number to the power 0 is 1” or “It’s just a definition.” Their answer did not stem from an extension of definition based on mathematical thinking but from a mathematical fact they remembered.

Lastly, the teachers whose responses were classified as “it cannot be defined” are the ones who adduced a reason for why it cannot be defined. The reasons were either the exponent rule and properties or the approach of a limit of a function. The teacher whose explanation is indicated in Figure 3 understood the exponent rules and properties well and explained
that $0^0$ cannot be defined because $0^0 = 0 \div 0$ while extending the base to 0, and dividing by 0 is not defined.

Figure 3. Teacher’s answer to $0^0$

Most of the teachers who concluded “it cannot be defined” compared the results of extending the property “$a^0 = 1$ where $a$ is a non-zero real number” to the case of 0 and extending the property “$0^a = 1$ where $a$ is non-zero real number” to the case of 0. They explained that as the results are 1 and 0, it is better not to define $0^0$. A similar response was obtained in Borasi’s study, where he set up these two alternative patterns that would lead the students to consider the plausibility of either 1 or 0 as possible values for $0^0$ and then presented them with the resulting contradiction (which in turn could make one consider either solution as an “error”). He suggested that some mathematical errors have the potential to stimulate students’ reflections on the nature of mathematics and make them realize that mathematics, too, is the product of human activities and thus not always as “perfect” as they might have expected.

On the other hand, some of teachers justified it using a limit of a function instead of exponent rules and properties. As indicated in Figure 4, one explained that “the limit of the function $f(x) = x^x$ can be 1 or some other numbers. So, we cannot define it.”

Figure 4. Teacher’s answer of $0^0$

The explanation states that we cannot define the value of the function continuously at 0 because the limit of the function differs according to the directions of $x$. This teacher may have treated the limitation $\lim_{x,y \to 0} y^x$
considering the base $x$ and the exponent $x$ as different variables and then explained that the two-variable function $f(x, y) = y^x$ cannot be defined continuously at $(0, 0)$. Even though the teacher did not use school mathematics, it is significant that the teacher tried to find the most natural way to define, extending the definition to university mathematics, consistent with what he/she already knew.

The teachers’ responses to the question on $\gcd(0,0)$ are classified into the following three types.

- I don’t know.
- 0.
- It cannot be defined.

Some of the teachers who responded as “I don’t know” said that they “hadn’t thought about it before.” Additionally, they justified not trying to extend the definition by saying “For divisors, only natural numbers are considered. But why $\gcd(0,0)$?” Mathematical definitions are not something fixed and unchangeable. Even though mathematical definitions and concepts are changeable as products of human history, these teachers did not consider the given definition or concept to be changeable.

The teacher who answered “0” said, “The greatest common divisor of oneself is itself, so $\gcd(0,0)$ equals 0.” and then applied this property of the greatest common divisor to 0, but did not apply the definition of the common divisor. The teacher who answered “It cannot be defined” explained (wrongly), “0 is a divisor of whole numbers. $(0, 0)$ cannot be defined.” One explanation is illustrated in Figure 5. The teacher tried to find the greatest common divisor by applying the definition of the common divisor to 0.

![Figure 5. Teachers’ writing about $\gcd(0,0)$](image)
Another teacher tried to find \( \gcd(0,0) \) by maintaining the property of the greatest common divisor that the teacher had learned in the university. This is shown in Figure 6. Even though the teacher made an error in calculations, he/she was reminded of the concept in number theory to extend the definition. The teacher said,

Teacher 8: \( \gcd(0,0) \) cannot be defined by the definition that satisfies the theorem used in number theory.

![Figure 6. Teacher's writing about \( \gcd(0,0) \)](image)

Teacher 8 applied the theorem she learned in number theory; Let \( a, b \in \mathbb{N} \) and \( a < b \) then \( \gcd(a, b) = \gcd(a, b - a) \). However, there were some errors in her answer. First, she did not consider the size of two numbers \( a \) and \( b \). Second, applying conclusion respectively \( a \) and \( b \), drew an improper conclusion; \( \gcd(a, b) = \gcd(a - b, b - a) \). It enables to extend the first result, rigorous understanding of mathematical theorems explained above. This teacher could not avoid the error since the teacher had deficient understanding about hypothesis and conclusion in the theorem.

\( 0^0 \) and \( \gcd(0,0) \) are not abstracted from concrete objects but are questions raised from possibility of their existences. When one extends an algebraic or geometric structure, it complies with the “principle of the permanence of equivalent forms” which means that the properties of the existing system must be maintained. In the given task, one cannot define the new mathematical object as something that maintains the operation structures and the definitions of “exponent rules and properties” and “common divisor of two integers.” We therefore do not define them.
Extending mathematical definitions helps not only in understanding mathematical theorems but also in understanding the mathematical properties and the structure of the concept. The mathematical concepts learned by students almost always appeared in a perfect form. The history of the concept such as the errors and epistemological difficulties faced while articulating the concept were excluded. Students find it difficult to accept mathematical concepts as consensual objects. However, defining activities provides students with opportunities to think that mathematical definitions are made by humans. It may convey to the students that they can be the subject who is defining mathematical concept. Furthermore, students can experience the principles followed by mathematicians for defining.

In the study of Rasmussen, Zandieh, King and Teppo (2013) on “advancing mathematical thinking” based on situated cognition, mathematical activities were identified in the form of symbolizing, algorithmatizing and defining during vertical and horizontal mathematization. In particular they explained that two different types of defining activities, descriptive and constructive, are identified in mathematics by Freudenthal (1973). Vertical mathematizing occurs in constructive defining and creates new objects by building on and extending these known objects.

The tasks proposed in the study, $0^0, \gcd(0,0)$ are a simple form of possible constructive defining. Rasmussen et al. (2013) developed their idea based on the perspective of mathematical thinking as acts of participation in a variety of different socially or culturally situated mathematical practices. They explored the process of generalization and abstraction in students’ discussion and arguments. Our study however explores individual cognitive subjects’ thinking process involved in defining.

3.3 A rigorous understanding of mathematical definition

Polynomial factorization is a basic skill required of middle school students. The following problem was developed on the grounds of teachers’ arguments on the correct answer to “Factorize $2x^2 + 4x – 6.$” The argument was whether to award full marks to students whose answers
factorized to \((x - 1)(2x + 6)\), or only accept fully correct \(2(x - 1)(x + 3)\) where 2 is factored out.

The question, shown in Figure 7, was developed to observe how teachers deal with the given constant polynomial “2” in the integral polynomial ring and in the real number polynomial ring, and on what grounds they make such judgments. A combination of mathematical concepts such as the factor, irreducible polynomial, and unit in the polynomial ring is involved in solving this problem. Hence, the question aims to check how university mathematics knowledge is used in understanding school mathematics.

**The following problems are related to factorization.**

1) Considering the polynomial \(2x^2 + 4x - 6\) as an element of the polynomial ring \(\mathbb{Z}[x]\), where the coefficients are integers, factorize it. If we consider this polynomial as an element of the polynomial ring \(\mathbb{R}[x]\) of which the coefficients are real numbers, does the result of factorization change?

2) If a middle school student has asked you to what extent he or she needs to factorize this polynomial, how would you explain it?

*Figure 7. Task for a rigorous understanding of mathematical definitions*

Particularly, the polynomial rings whose coefficients are integers, rational numbers, and real numbers handled in school mathematics are unique factorization domains (UFD). The above question is closely related to understanding the meaning of the uniqueness of UFD. As presented in the definition, “unique up to associates” means that which polynomial becomes the unit in the polynomial ring is important.

---

1 An integral domain \(R\) is a unique factorization domain (UFD) provided that every nonzero, non-unit element of \(R\) is the product of irreducible elements, and this factorization is *unique up to associates*; that is if

\[ p_1p_2 \cdots p_r = q_1q_2 \cdots q_s \]

with each \(p_i\) and \(q_j\) irreducible, then \(r = s\) and, after reordering and relabelling if necessary

\(p_i\) is an associate of \(q_i\) for \(i = 1, 2, \cdots, r\) (Hungerford, 1997, p.300)
Let $R$ be an integral domain. Then $f(x)$ is a unit in $R[x]$ if and only if $f(x)$ is a constant polynomial that is a unit in $R$. The units in $R[x]$ are the units in $R$, and the irreducible constant polynomial in $R[x]$ are the irreducible elements of $R$. For example the units of $\mathbb{Z}[x]$ are $\pm 1$. The constant polynomial 3 is irreducible in $\mathbb{Z}[x]$ even though it is a unit in $\mathbb{Q}[x]$. Therefore in (1) the factorization should be $2(x - 1)(x + 3)$ since “2” is an irreducible constant polynomial in $\mathbb{Z}[x]$, and it can be factorized as any of $(x - 1)(2x + 6), (2x - 2)(x + 3), 2(x - 1)(x + 3)$, as “2” is a unit in $\mathbb{R}[x]$.

Teachers who said that the factorizations are the same in the two polynomial rings judged that “2” does not affect the results of the factorization.

The results of factorization are not different in Figure 8, where the teacher on the left wrote $2(x - 1)(x + 3)$ and the teacher on the right wrote $(x - 1)(2x + 6)$. During the interview, the teacher whose response is indicated on the left explained why the factorization is the same in the two polynomial rings,

Teacher 5: 2 is a constant in $\mathbb{Z}[x], \mathbb{R}[x]$. So it does not matter when it is taken out as a common factor or not, and the result does not change in $\mathbb{Z}[x], \mathbb{R}[x]$.

These teachers could not find out that the question relates to UFD in algebra. Concepts from university mathematics were not involved. The second teacher, who gave similar reasons for no difference in factorization in the two polynomial rings, answered that only $Z(x - 1)(x + 3)$ can be accepted as correct when assessing students.
Teacher 7: School exams test how well a student has learned; therefore, he or she should be taught to factor out integers. The fundamentals of factorization involve factoring out ‘m’ from ‘ma+mb’. The reasons for not factoring out 2 here are the lack of understanding and the rule made at school.

The teacher, who said that the factorizations are different in the two polynomial rings explained that “2” is a common factor in an integer polynomial ring, but it can be considered as factored even when “2” is not factored out separately in the real number polynomial ring.

The first teacher whose response is indicated in Figure 9 had judged “2” as a unit in the integer polynomial ring and real number field, and this affected the factorizations in the two polynomial rings. Mentioning the need to know what is meant by a polynomial being uniquely factorized in this question, the teacher explained,

Teacher 1: It is asking about the unique factorization in the polynomial ring. It is saying that it is unique considering that it is the same as
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Multiplying a unit on the unique prime factorization. (...) 2 is not a unit when the coefficient is an integer. (...) Considering the coefficient to be real number, 2 is a unit. (...) Then it does not matter whether it is included or not. So the result should be different.

To the second question, the teacher answered “Our middle school curriculum dealt with the factorization in \( \mathbb{Q}[x] \). The teacher also said that “2” is a unit in the rational number polynomial ring covered in school mathematics, and hence whether or not it is factored out was not problematic.

The question required a correct understanding of the meaning of uniqueness in UFD. Precise understanding of mathematical definitions clarifies the range of mathematics being handled in school mathematics. Moreover, it helps us grasp the positioning of the contents of school mathematics within the structure of mathematics and the extent to which it has been simplified. It is important in that it enables elementary school mathematics to be taught without contradicting the advanced mathematical thinking that students will experience in the future. The three given tasks were related to analysis, number theory, and abstract algebra. Concretely, infinite sequence task for rigorous understanding of mathematical theorems was related to the properties of convergence sequence. Defining task for understanding of extension of mathematical definitions was related to exponential theorems and the properties learned in number theory. The last factorization task for rigorous understanding of mathematical definitions was related to UFD learned in abstract algebra, not requiring more than the mathematical knowledge learned in university. Yet, it provides how mathematical concepts like definitions and theorems learnt is utilized, and what sort of understanding must be further considered in the learning process of university mathematics in order to do so.

4 Conclusion

This study explored how teachers can combine university mathematics and school mathematics in order to embody their knowledge in practice,
and the kind of supplementary understanding needed in teacher education institutes. In other words, teachers need to transform the university mathematics instead of delivering it as it is. The following are the implications of what is required to help the teachers in implementing their experience of advanced mathematics.

- A rigorous understanding of mathematical theorems,
- An understanding of extension of mathematical definitions, and
- A rigorous understanding of mathematical definitions.

The findings of this study suggest that expanding a subject-matter knowledge of university mathematics such as algebra and analysis is not a solution. It emphasizes the behavioural aspects that can help future teachers while they learn university mathematics. It focuses on knowing as a ‘process’ of getting knowledge.

The problem of the first task can be given to students who learned convergence of infinite sequence, and the given solution contains errors that students could make while solving the problem. It is a necessary knowledge for teachers to find out errors in students’ solution. At that time, teachers need rigorous understanding of mathematical theorems. In second tasks, teachers need to understand not only the mathematical properties, the structure of the concept, but also principle of the permanence of equivalent forms. An understanding how to extend mathematical definitions, it can play an important role to connect university mathematics and school mathematics, because it helps to understand how the concept in school mathematics is related to university mathematics. The last factorization task shows that a rigorous understanding of mathematical definition helps us grasp the positioning of the contents of school mathematics within the structure of mathematics and the extent to which it has been simplified.

Taking a greater number of courses related to CK or PCK does not ensure the prospective teacher’s competency (Monk, 1994). More importantly, future teachers need to understand mathematical definitions more precisely in order to know the role of the concepts of school mathematics in the formalized mathematical structure. They also need to
learn, through defining activities, how the new concepts can be extended based on the contents of school mathematics. Moreover, they need to know how precisely defined mathematical concepts are transformed in school mathematics.

This process is expected to empower mathematical thinking including the practice of applications as well as usage of mathematical language, skills, and definitions. This study explored teacher knowledge to enhance future teachers’ mathematical empowerment. Ernest (2002) distinguished empowerment as three types; separated social empowerment, epistemological empowerment, and mathematical empowerment. Social empowerment refers to the ability of using mathematics to reconsider one’s opportunity in education and career. Epistemological empowerment refers to high personal confidence in not only mathematics but also the creation of knowledge and its utility. To solve the discontinuity of school mathematics and university mathematics, this study explored empowerment in the narrow context of prospective teachers’ university mathematics, i.e. mathematical empowerment.

Assessment questions developed by MKT researchers are actively studied mostly in the context of elementary school teachers’ knowledge. MT21 and TEDS-M have been developed by researchers based on secondary school teachers’ knowledge. However, there has been an inadequacy of assessment questions examining university level mathematics that is closely related to school mathematics. The PCK assessment questions demands low-level mathematical knowledge and the CK assessment questions are not significantly different from school mathematics. Since the questions in this research were developed with a view of enabling the use of university mathematics to deal with school mathematics, it was possible to examine university-level mathematical knowledge as well as the content knowledge of school mathematics and PCK. The assessment questions developed in this study overcame the lack of content knowledge of advanced mathematics in the assessment questions of other studies. It is expected that the questions can be used to evaluate teachers’ knowledge.

The study also presented specific examples about the effect of advanced mathematics on teaching school mathematics. This data is
relevant as it demonstrated that abstract algebra, particularly UFD, affects teachers’ understanding of factorization in school mathematics and that theorems used for analysis can be discussed in the context of school mathematics.

Experiencing advanced mathematics, which is precise and abstract, is not aimed at the quantitative expansion of mathematical knowledge. It aims to develop the ability to coordinate and reconstruct content in order to gain mathematical thinking ability. In this regard, this study ascertained the types of activities that must be emphasized to cultivate educational capacity while learning university mathematics. This result can be used to enhance the expertise of not only future teachers but also in-service teachers.

As mentioned before, SRCK is not conceptualized based on experiences in the teachers’ practice. It is currently at the stage of theoretical conceptualization. Whether any type of relationship exists between SRCK and other domains of MKT is controversial. It remains difficult to clarify its position with respect to our research here. Even though this study explained the necessity of three types of understanding on university mathematics learning, it did not identify the method for enhancing the three types of understanding. It can be taken up by future research.

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References


Student voice in the mathematics classroom can be interpreted in two distinct but interconnected ways. Firstly, we could think of giving voice to students through occasioning opportunities for students to be actively engaged in mathematical discussions and mathematical argumentation—both in group and whole class settings. Secondly, we can think of student voice in terms of developing student agency—agency that supports students to feel in control and empowered in their learning. In this chapter, we look at three instructional practices that enable teachers to support the development of student voice in the mathematics classroom. Embracing the duality of meaning, we look at (i) how teachers can support students to engage in rich mathematical discourse through group activities, (ii) how teachers who notice and value their students’ thinking use this thinking as a resource for learning, and (iii) how teachers can actively position their students as competent. In combination, these practices serve to empower students as active learners of mathematics.

1 Introduction

In looking to understand how we might best empower mathematics learners, we first need to understand what empowerment means within the mathematics classroom, the mathematics curriculum, and for individual learners. We need look no further than the Association of Mathematics
Educators (AME) 2016 yearbook (Toh & Kaur, 2016) to see numerous references to 21st century competencies concerning confidence, self-direction, collaboration, innovative problem solving, and citizenship (Ministry of Education, 2010)—all components of an empowered mathematics learner. In this chapter, we look at how developing student voice in the mathematics classroom contributes to these 21st century competencies.

The enabling and exercising of student voice is an essential part of the social dimension of learning and communicating mathematics. Drawing on social learning theory (Wenger, 1999) student voice is implicated in the learners’ construction of competence, the formation of identity and dispositions, and the development of participatory and mathematical practices (Anthony & Hunter, 2010). While mathematics education researchers note differences in the way student voice is occasioned in classrooms around the world (see Kaur, Anthony, Ohtani, & Clarke, 2013), there is increasing consensus that productive mathematical discourse within group work and class discussions serve to support students’ sense-making and problem solving (Hino, 2016; Kaur, Wong, & Bhardwaj, 2016; Koay, 2016; Wagganer, 2015; Walshaw & Anthony, 2008).

In this chapter we interpret student voice in two distinct but interconnected ways. Firstly, we think of giving voice to students within opportunities for students to be actively engaged in mathematical discussions and mathematical argumentation—both in group and whole class settings. Secondly, we think of student voice in terms of developing student agency—agency that supports students to feel in control and empowered in their learning. Integrating both these perspectives, we look at three instructional practices that enable teachers to support the development of student voice in the mathematics classroom. We look at how teachers can: (i) support students to engage in mathematical discourse within collaborative group work, (ii) notice and value their students’ mathematical thinking, and (iii) actively position their students as competent. We show how when used in combination, these instructional supports serve to empower students as active learners of mathematics.
2 Engagement in Mathematical Discourse

Because learning mathematics is an inherently social process (Jorgensen, 2014) our pedagogical approach should capitalize on the affordances of social interactions involving talk. Mathematical talk is valuable on so many levels. For example, mathematical talk:

- helps students to clarify and organise their thoughts
- facilitates personal and collective sense making
- supports building connections between representations and multiple strategies
- enables students to use others as a resource of ideas to challenge and broaden understanding
- helps students learn mathematical language
- enables the sense of authority to move from teacher to discipline
- provides a resource for teachers to build on their thinking
- supports the development of mathematical identity
- enables students to see mathematics as created by communities of learners

Mathematical talk can occur between multiple participants and in multiple situations within a lesson. In particular, cognitively challenging problem-solving tasks afford rich opportunities for mathematical talk (Anthony & Hunter, 2010). Collaborative problem-solving tasks provide opportunities for students to voice their diversity of ideas, to develop skills in mathematical argumentation, and to develop prosocial skills of cooperation and respect. Moreover, when the tasks are ‘open-middled’ or ‘open-ended’ (Sullivan et al. 2015) students can experience multiple and multidimensional paths to problem solving. This means that students are able to enter, and sometimes to exit the problem, in ways that match their thinking. Problem tasks also enable student opportunities to interact with a variety of alternative conceptual systems (Lesh & Zawojewski, 2007). For example, in the following activity (Figure 1) students can solve this task by using either an arithmetic, geometric, or conceptual approach (see http://nzmaths.co.nz/level-5-rich-learning-activities).
Activity:

A car rental business has two rental schemes, red and blue for rentals up to ten days. These schemes are advertised in their brochures with this graph.

Use the following information to work out how many days of rental would carry the same total cost on either the blue or the red scheme.

The area under the graph gives the total cost of renting.

The schemes each follow a linear pattern, cutting the vertical axis at 120 and 170.

Both blue and red schemes cost $110 on the 4th day of rental.

Being accountable requires establishing norms so that when each person’s ideas—be it student’s or teacher’s—are voiced they are taken seriously, with all contributions equally valued. In Hunter’s (2007) study of primary-level classes, expectations were most effective when they were co-negotiated with students. Examples of negotiated rights and responsibilities included: the right of all students to contribute and to be listened to; the right to test out ideas that may or may not be correct without fear of having other students making disrespectful comments; and the right to have other people discuss your ideas rather than you personally.

In establishing these norms, the teacher needs to take care to ensure that mathematical talk is not just for those students with strong verbal skills or who are confident about speaking out. For some learners, there are individuals or groups who are likely to be more comfortable in the role of listening respectfully to the teacher. In our research classrooms, an example of proactive action taken by a teacher to a Pasifika girl’s contribution was: “You don’t have to whisper. You can talk because we want to make sure that you are heard.” On another occasion a student was told to “speak up, I like the way you are thinking but we need to hear you”. By regularly calling on students to respond to the ideas that are being discussed, regardless of whether they volunteered, the teacher reaffirms
that all students are expected to make sense of these problems and all students are expected to participate. In addition, in collaborative problem-solving tasks involving solutions based on the contributions of the group members, it is important that there is space to affirm and value individual contributions. A student in our study noted that when the group is working well “we think about all the different ways before a decision is made about the group’s strategy solution”.

Establishing group norms and roles takes time. A particularly challenging role when working in groups is the need to ask mathematical questions of one another and actively listen to each other’s answers. Students first learn to do this by interacting in discussions with the teacher that model the use of questions which clarify or extend aspects of an explanation. For example, students can be pressed to explain their solution strategies with questions starters like: What did you do there? Where did you get that number 6 from? Can you show, draw, or use materials to illustrate what you did? These questions can then be combined with prompts to offer justifications associated with agreement or disagreement such as: Why did you…? So what happens if…? But how do you know it works? Can you convince us? So why is it that…? This modelling is often accompanied by regular affirmation of group participatory norms. For example, in the following episode we see how a teacher, reflecting the Pāsifika values of her student group, reinforced her students’ personal responsibility to question and sense make:

Make sure you share to the whole group. If you do not understand then remember you need to ask questions and if someone in your group doesn’t understand, you need to help that person or you ask them do you understand. Remember we work like whānau [family], that’s our way of working. We are all taking a risk and if you try your best to say something that’s okay. Take a risk to say something or ask a question and then we’ll help you.

The following episode illustrates how students can adopt question prompts in their group activities to develop shared understanding of all the group members:
Aroha: [records 43, 23, 13, 3 and then $3 \times 4 = 12$] I am adding forty-three, twenty-three, thirteen, and three, so three times four equals twelve.

Kea: What are you trying to do with those numbers? Where did you get the four?

Donald: All she is doing is like making it shorter by like doing four times three.

Hone: Because there are only the tens left.

Donald: Three times four equal twelve and she got that off all the three, like the forty-three, twenty-three, thirteen, and three. So she is just like adding the threes all up and that equals twelve.

When involving students in responding to teachers’ questions, or questioning others, it is important that students are held accountable to listen and make sense of the mathematical explanations. Questions that support students to make connections between mathematical ideas and test generalizations include: Does that always work? Can you give us a similar example? Is it always true? Do you agree or disagree? Can you link all the ideas you have used? Again, these more sophisticated ways of engaging in mathematical argumentation require consolidation of expectations around participation. The following episode illustrates how a teacher reminded her students of their role in sense-making:

[In your groups you need to]…talk about what you are doing…so whatever numbers you have chosen don’t just write them. You say, I am going to work with…or I have chosen this and this because…and this is what I am going to do…you need to explain how you are working it out to your group. They are going to listen. I want you to think about and explain what steps you are doing, what maths thinking you are using. The others in the group need to listen carefully and stop you and question any time or at any point where they can’t track what you are saying.

Establishing group norms for communication and participation and modelling discourse patterns are important ways that teachers can support students to become proficient at mathematical justification while also
developing students’ awareness (metacognitive knowledge) that learning involves listening and engaging with other people’s ideas.

3 Noticing Students’ Mathematical Thinking

Noticing and making sense of student thinking is considered essential for effective teaching (Miller, 2011). In mathematics education, teacher noticing actions have been various described as attending, interpreting, and deciding (Jacobs & Empson, 2016), eliciting, supporting, and extending (Fraivillig, 2001) or eliciting and interpreting students’ mathematical thinking (Sleep & Boerst, 2012). Underlying each of these frameworks is the assumption that we need to create learning opportunities based on what students know and need to learn. That is, when teachers attend closely to students’ thinking they will be in a better position to provide tasks/questions that are appropriate to students’ engagement with the mathematical concepts.

Occasioning multiple opportunities for students to voice their thinking supports teachers to engage in professional noticing. Researchers (e.g., Chapin & O’Connor, 2007, Kazemi & Hintz, 2014) studying discourse rich classrooms have categorised a number of ‘talk moves’ that teachers can use to facilitate or orchestrate mathematical talk as follows:

- Revoicing: “So you are saying…”
- Repeating: “Can you repeat what she said in your own words?”
- Reasoning: “Do you agree or disagree and why?”
- Adding on: “Would someone like to add on to this”?
- Wait time: “Take your time…”
- Turn-and talk: Turn and talk to your neighbour…”
- Revise: “Has anyone’s thinking changed?”

The following episode (from Anthony, Hunter, Hunter, & Duncan, 2015) involving a Quick Image activity (Figure 2) illustrates how teacher talk moves can support students to explain their reasoning and encourage them to engage with each other’s ideas through active listening, agreeing or disagreeing, or adding on:
Hamuera: I saw that you could count in threes.
Teacher: You could count in threes, that’s interesting, what do you mean by that Hamuera?
Hamuera: Like 3, 6, 9.
Teacher: Oh interesting, so you are skip counting in threes?

As the lesson proceeds, we see how the teacher continued to value and attend to students’ thinking. As each student voiced their solution strategy the teacher created a parallel representation of the dots on the blackboard. In inviting some students to ‘demonstrate’ their thinking with reference to the representations on the whiteboard, the teacher expertly provided space for the students to further unpack their thinking:

Teacher: Who else counted in groups of three? Lots of people. Carol can you show us your groups of three that you counted in, come up and point them out for us?
Carol: [approaches the whiteboard and points to the top dot of each vertical grouping of three]
Teacher: So what was this one, what did you say in your head when you looked at them?
Carol: I counted three of those, three of those, three of those [gesturing vertically for the 3 x 1 and 3 x 2 arrays] and then
three of those, and three of those, and three of those [gesturing horizontally for the 3 x 3 array].

Through observing students at work on group tasks, teachers can also use talk moves to press students to explain their thinking in ways that add to mathematical meaning. In the following example, involving the problem of sharing three cakes between eight people, the teacher intercedes in a group discussion to press students to provide conceptual rather than procedural explanations. She does this by listening and responding to Hemi’s mathematical thinking in a way that directs students’ attention to very specific aspects of their explanation [what does the twenty four mean?]:

Hone: What are you doing?
Anaru: Twenty four eighths.
Teacher: But I am not sure…we know what you mean. Can you explain it?
Anaru: [Frowned and shook her head in response]
Hemi: [points at the symbols 24/8 which Anaru had recorded next to the drawing] I can. Twenty four eighths, because there are eight in each cake and there’s eight slices in each cake and it all adds up to twenty four.
Teacher: Twenty four what? What does that bottom number mean?
Hemi: That means how much slices in each cake.
Teacher: Okay, what does twenty four represent?
Hemi: It means how much altogether.
Teacher: Altogether, Yeah, twenty four bits, slices and they are all eighths.

One way in which teachers can prepare themselves to make sense of student thinking is through the thoughtful anticipation of students’ mathematical responses (Stein, Engle, Smith, & Hughes, 2008). The practice of considering student thinking during lesson preparation has been shown to influence how teachers make sense of and use student thinking in the enactment of lessons. As Bray (2011) notes those teachers with “sound knowledge are more apt to notice and respond to critical
learning moments in the lesson, and are more likely to be able to use students’ thinking as springboards for inquiry in the context of class discussions” (p. 35).

Professional noticing also requires that we notice who is participating, whose voice is heard and whose is not. For students to engage in mathematical talk they need to feel able to take intellectual risks. In the Quick Image episode above (Figure 2) the teacher chose to accept incorrect answers alongside correct answers as a starting point for discussion. Initially the teacher made no evaluation as to the correctness of students’ solutions, preferring to use the ensuing discussion explore why certain aspects of the solution process were correct or not correct. By deliberately choosing to focus publically on what students can do, as opposed to what they can’t, the students are affirmed in their role as risk takers and learners.

4 Positioning Students as Competent

To support equitable participation, teachers need to capitalize on the multiple ways that students can be positioned as competent (Hand, Kirtley, & Matassa, 2015). A first step is to provide sufficient ‘think (wait) time’ in situations where one expects students to offer conjectures, respond to questions, or examine the thinking of others. Providing explicit thinking time sends a clear message that students are expected to engage with the task in ways that require effort and time to get one’s thought organized. For example, in seeing a student struggling to develop an explanation the teacher in one of our research classrooms offered wait time as follows: “I can see you are confused. Me too, that’s all right we can take some time to rethink about it. It’s good to take some risks with our thinking sometimes.” In this way, the teacher’s validation of the acceptability of confusion as a natural part of sense-making served to normalize that learning mathematics involves effort and sometimes struggle. Moreover, the consistent use of thinking time helps to disrupt the finishing first or just knowing the answer signifiers of ‘being good’ in mathematics (Kazemi & Hintz, 2014).
Another way to assign competence, especially when students are working in groups, is to take a proactive role in highlighting contributions from less able or less vocal students (Boaler, 2008). For example, when a teacher in our study (Hunter & Anthony, 2011) noticed a valid contribution by a low achieving student she commented: “Wow, Teremoana see how you have made them think when you said that? Now they are using your thinking.” More than just positioning Teremoana as someone with good ideas, this action modeled the teacher’s appreciation of diversity in students’ contributions.

Appreciating the diversity of voices within discussions necessitates that both correct and partial or flawed thinking will be shared. For many teachers, responding to student errors can be a source of tension; they need to decide in the moment whether to take up or ignore the contributions or how to work with the student’s meaning (Brodie, 2010). However, rather than be seen as a negative, incomplete thinking and mathematical errors can provide a valuable window into students’ thinking—a dynamic assessment opportunity (Storeygard, Hamm, & Twomey Fosnot, 2010). The following episode illustrates how a teacher worked with unanticipated responses from Rio and Sose; responses that indicated confusion between groups of three and three groups. In this episode we see how the teacher, rather than use her authority to show the correct answer, chose to draw on a wider range of students’ thinking to resolve the difficulties. We re-enter the Quick Image lesson discussed earlier (see Figure 2) with the teacher prompting students to link the total number of groups of three to the notation of $3 + 6 = 9$ and $9 + 9 = 18$:

Teacher: Can you see how many groups of three are over here? [pointing to $3 + 6 = 9$ and $9 + 9 = 18$] Rio?
Rio: Three.
Teacher: Where are your three groups of three, can you show us? [blank look from Rio]
Teacher: How many groups Sose, of three, do we have here? [pointing to the $3 \times 1$ array]
Sose: Three [hesitantly]?
Teacher: How many GROUPS of three, in just this part here, what do you think? In just this part here, one, two, three [pointing to each dot in the 3x1 array] how many groups of three?

Sose: Three.

Teacher: Okay, why do you think it is three?

Sose: Cause there are only three dots.

Teacher: Cause there are three dots, okay that’s cool. Let’s have a look [draws a 3x1 array on the board]

Teacher: Brittany how many groups of three can you see in that first part?

Brittany: I can see one group of three.

Teacher: Sose can you see where Brittany says she has her one group of three? Brittany can you go up and show us [at the whiteboard, Brittany points to the first group touching the bottom dot].

Teacher: Is it just that bottom one, where is the group of three? [Brittany re-gestures to include the whole 3 x 1 array.] So how many groups of three have we got all together?

Brittany: Nine.

Teacher: Nine groups of three, where are they, show us [pointing to the 3 x 3 array]. Is that nine groups of three?

Brittany: Three together, all together it makes nine.

As the discussion proceeds we see how the exploration widens to include a range of peers’ contributions rather than reliance on the teacher’s own ‘retelling’ of how many groups of three can be seen. Having a correct interpretation of the number of groups of three in the mix, the teacher then took care to return to Sose—sending a clear message that the discussion is multi-directional, that it involves the thinking of the collective, and that as a teacher she cares about how they are coming to understand mathematics.

As well as providing opportunities for all students to provide input into solutions, to have their voice heard and valued, it is also important that all students having access to cognitively demanding tasks. Cognitively demanding tasks afford opportunities for each student to struggle and respond to mathematical challenge in ways that support their learning of big mathematical ideas. However, teachers often worry that tasks might be too difficult for students who struggle, and sometimes unwittingly lower
Developing Student Voice

One way to ensure that challenging tasks (see an example in Figure 3) are accessible to a wide range of students is to use tasks that have more than one entry point. Multiple entry points typically allow for learners to access and use multiple means of representations (e.g., visual aids, tables, materials), and experience multiple means of engagement (e.g., working independently, in pair-share group, whole-group discussions, and whole class discussions).

In addition, Sullivan et al. (2015) suggest the use of enabling prompts as a specific strategy to ensure that these tasks allow all students to use their own way of solving the problem, particularly those students who you might expect to experience difficulties. Enabling prompts can work in multiple ways. For example, the enabling prompt might reduce the number of steps in the problem solution, simplify the complexity of the numbers, or vary the forms of representations. In the example above (Figure 3), an enabling prompt could take the form of a simpler question such as: ‘\_ \_ \times 3 \_ = \_ \_ \_ 0’ What might the digits be?’ Scaffolding the student in this way means that the student can then proceed with the original learning task. As Sullivan and colleagues note an important feature of this approach is that students can “feel part of the classroom community and the lesson in that they have created knowledge for themselves” (p. 126). To extend the thinking of those students who complete the learning task quickly, Sullivan and colleagues recommend that teachers plan extending prompts. Such prompts might involve, for example, a press to “seek abstractions and generalization of solutions or some aspect of proof associated with the completeness of answers or legitimacy of the solution strategy” (p. 126).
Collectively, these prompts affirm expectations that students themselves can keep thinking and learning and thus this teacher action serves to develop productive notions of competence.

5 Implications and Conclusion

In this chapter we have argued that encouraging and supporting student voice is a key facet of empowering mathematics learning. Indeed we go further and argue that providing opportunities for students to engage in mathematical talk is central to the being part of a mathematical inquiry community. Each of the preceding sections provides examples of ways that teacher can develop a classroom climate and activities that encourage and promote student voice as an integral part of learning mathematics.

We take care to note that having students engage in rich mathematical talk requires careful consideration of norms of participation that are built on principles of equitable and respectful collaboration. Moreover, students need to be supported to develop mathematical practices associated with mathematical argumentation. In particular, support can come in the form of collaborative problem-solving tasks that provide opportunities to engage in public mathematical exchanges.

Providing appropriate challenge involves teachers planning and designing tasks that allow all learners to “build from their own thinking and access the thinking of their peers to bridge new concepts” (Lambert & Stylianou, 2013, p. 503). To this end, enabling and extending prompts that support all students to engage with the big mathematical ideas within a task form an important part of task planning (Sullivan et al., 2015).

The teachers’ role in orchestrating classroom discourse is critical. Teachers need to balance their inherent position of authority with the development of student agency. Public discourse exchanges must support student risk taking, promoting relational equity (e.g., Boaler, 2008) in ways that value of the diversity of students’ different ways of thinking and different viewpoints. Classroom cultures that value learning from mistakes, that respect students who offer errors for discussion, and where opportunities to revise one’s thinking is commonplace, will help build productive student identities (Bray, 2013). In noticing and valuing their
students’ thinking and reasoning, teachers can use this information to maximize the mathematical potential of their students. Taking care to position each student as competent affirms their identities as mathematic learner.

References


Chapter 7

Empowering Mathematics Learners through Effective Memory Strategies

WONG Khoon Yoong

Mathematics learners must remember different types of information, such as the meanings and examples of concepts, standard procedures, and justifications of mathematics principles. Those who fail to do so effectively may forget what they have learned, recall flawed procedures, or confuse similarly looking formulae, such as \(2\pi r\) and \(\pi r^2\), through memory decay or retroactive interference. This chapter proposes a 5-stage information processing model and explicates several strategies to help students overcome the memory problem. This framework covers ways of acquiring, processing, encoding, strengthening, and retrieving information encountered during mathematics learning. Students can be empowered to develop meta-memory about how memory works and to apply effective memory strategies on their own to enhance their mathematical performance.

*Memory is the mother of all wisdom.* Aeschylus, an ancient Greek playwright (525–456 BC)

1 Introduction: The Memory Issue

Consider this common scenario. The students listen attentively to an explanation of how to add fractions using the standard algorithm. They practise this method with a few examples with some success. A few days

1 http://www.brainyquote.com/quotes/authors/a/aeschylus.html
later, they learn how to multiply fractions. After that, when they return to fraction addition, some of them add the numerators and the denominators separately, as if they were doing multiplication. Their subsequent learning has retroactively interfered with recall of previously learned materials. Of course, mathematics learning is more than just reproducing standard procedures; it includes meanings, justifications, reasoning, and problem solving. Nevertheless, for each type of learning, students may forget some of the things they have learned (memory decay) or become confused between new and old learnings (retroactive interference). This is an especially acute problem for low achieving students, who are often thought to have innate “weak memory.” This attribution is not helpful because it is difficult to change traits that are believed to be innate, leading to a sense of helplessness among teachers and students. On the other hand, recent advances in neuroscience and neuropsychology have found that memory is malleable throughout a person’s lifetime, and these findings suggest new ways to empower learners by developing effective memory strategies. This chapter provides a synthesis incorporating some of these new insights into a 5-stage information processing model applied to mathematics learning.

In recent years, numerous studies and books have been published about the nature of memory and the brain and the implications of neuroscience for education (Brown, Roediger, & McDaniel, 2014; Medina, 2008; Oakley, 2014; OECD, 2007; Sousa, 2015; Sprenger, 2010; The Royal Society, 2011; Zull, 2011). The brain has been characterised as a learning brain, a teaching brain (Rodriguez, 2014), an emotional brain (Immordino-Yang, 2016), and so forth. The 5-stage model below is my attempt to synthesise some significant ideas from these sources. Since the full model has not been verified by research, it is offered here as an attempt to uncover possibilities of using these ideas to benefit all mathematics learners whether they study offline or online (Miller, 2014). However, it is necessary to note at the outset a caution raised by Rose and Rose (2016): “[N]euroscience’s advice on how best to teach is unable to take account of the socioeconomic causes” (p. 110). These causes, such as poverty and lack of education resources, must be addressed separately to complement what educators can learn from neurosciences.
2 Why Memory Matters

Memory seems to have a negative reputation among some educators, but it is a critical factor in problem solving and long-term mastery of knowledge. For mathematics learning, the ability to recall important information matters because of the following considerations:

- Good memory ensures that routine procedures can be completed with speed and accuracy, resulting in enhanced performance. According to Miller (2014), memory “is the mechanism by which our teaching literally changes students’ minds and brains” (p. 88), and good teaching can bring about these changes.

- The ability to automatically recall standard information frees up space in the working memory so that the learner can handle higher order thinking with reduced anxiety. Reducing cognitive overload on working memory through automatic recall of standard results is one of the seven principles of mathematics teaching for mastery recommended by the National Centre for Excellence in the Teaching of Mathematics (NCETM) in the United Kingdom\(^2\). Standard mathematics results cover number facts, unit conversions, and basic formulae, and these are the necessary foundational knowledge for higher level learning.

- Looking up information from external sources such as notes and websites is often portrayed as essential 21st century skill, and it is recommended as an appropriate strategy, especially among Western educators (e.g., see a recent report in *The Telegraph*\(^3\)). However, doing so will use up precious mental resources, which can be put to better use during problem solving.

- Students who can demonstrate strong memory will have a sense of achievement and positive feelings about this achievement.

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\(^3\) [http://www.telegraph.co.uk/education/12079330/Learning-times-tables-is-unnecessary-because-children-can-look-up-answers-on-mobile-phones-says-union-leader.html](http://www.telegraph.co.uk/education/12079330/Learning-times-tables-is-unnecessary-because-children-can-look-up-answers-on-mobile-phones-says-union-leader.html)
What is memorised can be used to learn new information in a coherent way, connected to prior knowledge. According to Lang (2016), “[w]ithout any information readily available to us in our brains, we tend to see new facts (from our Google searches) in isolated, noncontextual ways that lead to shallow thinking” (p. 15).

Finally, memory begets stronger memory in the sense that the more a person can remember, the more he/she will remember new things. It is a skill that can be honed through deliberate practice (see Section 7.1 below).

The on-line Merriam-Webster Dictionary defines “memorize” as “learn (something) so well that you are able to remember it perfectly.” This definition does not imply that to remember something is achieved only by mechanical and joyless repetitions, which are often associated with “learn by rote” or “learn by heart.” To remember something is complex and active, which is captured below by a 5-stage information processing model derived from cognitive psychology.

3 The 5-Stage Information Processing Model

Information processing models describe the cognitive structures (memory systems) and cognitive processes undertaken when the human mind processes information during learning, thinking, and problem solving (see Cowan, 2014, for a historical review of several models). Using the computer as a metaphor, these models divide cognitive functioning into an input-processing-output sequence involving different types of memory (Anderson, 2010). Theories of memory postulate an encoding-storage-retrieval framework (Miller, 2014). Findings from cognitive psychology about how these memories work are now supported by studies in neuroscience about how the brain functions. Figure 1 provides a synthesis of these models, illustrated with an example about finding the area of a...
circle. The input is how to find the area of a circle, the processing involves five key stages, and a final output is produced. An “emotion gateway,” not found in standard information processing models, is included here, based on neurological findings about the intricate relationships between emotions and cognition. As pointed out by Medina (2008), “[t]he brain remembers the emotional components of an experience better than any other aspect” (p. 83). Indeed, both emotion and cognition must function together to ensure deep learning.

![Figure 1. Information processing model with emotion gateway](image)

The five stages are summarised in Table 1. Note that the last row about meta-memory is not part of the model; meta-memory strategies can be applied at each of the five stages of the model.
Table 1
The 5 stages of memory use and meta-memory

<table>
<thead>
<tr>
<th>Stages</th>
<th>Memory Use</th>
<th>Recommended Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Acquire information through different senses</td>
<td>Use multiple sensory modes to gain and sustain attention</td>
</tr>
<tr>
<td>2</td>
<td>Process information in working memory</td>
<td>Overcome emotional obstacles, use chunking, memory triggers, goal-free problems</td>
</tr>
<tr>
<td>3</td>
<td>Encode information into long-term memory</td>
<td>Stress intention to remember, build meaningful connections, use graphic organizers</td>
</tr>
<tr>
<td>4</td>
<td>Strengthen information in long-term memory</td>
<td>Conduct deliberate practice and regular quizzes, address primacy and recency effects</td>
</tr>
<tr>
<td>5</td>
<td>Retrieve information from long-term memory</td>
<td>Use retrieval cues, address emotional and environmental factors</td>
</tr>
<tr>
<td></td>
<td>Meta-memory; change student beliefs about memory</td>
<td>Identify and change student beliefs about memory, stay healthy</td>
</tr>
</tbody>
</table>

Before reading further, the reader is invited to try the following activity:
   a) Spend 2 minutes to remember as many points from Figure 1 and Table 1 as possible.
   b) Spend 2 minutes to write down as many points as one can recall from memory.
   c) Check how many points are recalled correctly.
   d) Reflect on the effectiveness of one’s approach to remember these points.

The above activity covers the “learn → rehearse → test → feedback → reflect” cycle required for successful learning. Successful recall depends on an intention to remember at the “learn → rehearse” phase. Without this conscious intention, people may come across much information but remember little. The cycle can be repeated if one wishes to recall more of
Empowering Mathematics Learners through Effective Memory Strategies

4 Stage 1: Acquire Incoming Information

Before learning can take place, students need to be able to acquire relevant information through various sensory modes. Three main sensory modes are used in mathematics lessons, collectively referred to as the VATK system:

- Visual mode through the visual cortex; e.g., read writing on the whiteboard, PowerPoint slides, and worksheets; observe teacher’s actions; the information is stored in the iconic memory.
- Auditory mode through the temporal lobe; e.g., listen to teacher’s explanations, question-and-answer, discussion; the information is stored in the echoic memory.
- Tactile and kinaesthetic mode through the motor cortex; e.g., work on concrete manipulatives; the information is stored in the haptic memory.

Researchers in learning styles often stress that, in the words of Dunn and Dunn (1987), “students learn best when instruction and learning context match their learning style” (p. 55). However, they do not need to learn only in their predominant learning style. Indeed, they should become versatile learners, capable of using different modes to process information. Students learn better when they use several sensory modes simultaneously to process the same information. This is stated as the “temporal contiguity principle” for online learning by Mayer and Moreno (2003). Furthermore, given about forty students in most classes in Singapore, it is not feasible to ensure this match in learning styles. Hence, a more pragmatic approach is to encourage students to use all the sensory modes during lessons. To ensure ready and accurate access to the information, the teacher must pay attention to mundane details:

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6 http://acronyms.thefreedictionary.com/VATK
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- Visual: Do not write at the bottom and sides of the whiteboard because it is not visible to all the students; adjust proper lighting so that projections from PowerPoint or the visualizer are visible to all.
- Auditory: Use proper volume, pace, tone, and pronunciation of mathematical terms, e.g., to differentiate between tens and tenths. Ask the whole class to read aloud key information, and ask some students to repeat what has been said in their own words.
- Kinaesthetic: Check that all the students can follow the steps of using concrete manipulatives through masterful demonstrations.

The first critical concern must be that students pay attention to the incoming information and capture it for subsequent processing. Sensory information is held in milliseconds, and it will decay rapidly if not attended to. To capture their attention, try the following:

- Students do not pay much attention to boring, repetitive utterances. Arouse their curiosity by presenting surprising information, real-life applications, and stories that match the students’ needs and mathematical experiences.
- Present typical mathematics contents with enthusiasm (see examples in Wong, 2014).
- Highlight key points by boxing, colouring, or underlining them.
- Mention explicitly what to pay attention to, for example, saying “pay attention to this because it is important.”
- Alert the class to an impending question that they have to think about, by saying “I am going to ask you a question” (pause) and then ask the question.

Attention span is the amount of time a person can concentrate on a task without becoming distracted. Students have short attention spans, lasting between 5 to 10 minutes. Change activities after every 10 minutes or so, so that the students will use different sensory modes; for example, passive listening followed by writing or hands-on activity. Minimise distractions, such as fanciful animations in PowerPoint slides, and classroom disturbances. It takes time for people to recover from distractions; for example, Powers (2010), writing about online distractions, suggested that
“recovering focus can take ten to twenty times the length of the interruption” (p. 59). Disruptions under poor classroom environments may take considerably longer and greater effort to recover from.

5 Stage 2: Process Information in Working Memory

5.1 Emotion gateway

Once captured, the sensory information is relayed by the thalamus simultaneously to the emotional part (amygdala) and the cognitive part (prefrontal cortex) of the brain, as shown in Figure 2. At this stage, past emotional experiences play an important role in influencing how the student might respond to new information in order to get past the emotion gateway. Three basic options are observed:

a) When the new information triggers positive prior experiences, the student is likely to engage with it.

b) If it triggers extreme stress and anxiety, the amygdala may short-circuit the path to cognition and the student is likely to give up.

c) If mild stress is triggered, the student might be prepared to invest the necessary effort into processing it, since findings such as the Yerkes-Dodson Law suggest that a certain level of stress or arousal can enhance performance. The student must realise that initial difficulty, with minor stress, is “desirable” for real learning (Miller, 2014).
5.2 Working memory capacity

The information is now available for processing in the working memory. Working memory in adults is believed to have a limited capacity of about $7 \pm 2$ isolated items, and probably less in students. Students can gain an idea of their working memory capacity by taking the test at http://cognitivefun.net/test/7. Practising with this test may enlarge one’s working memory capacity, but for real learning, it is more meaningful to convert isolated items of information into chunks, where a chunk consists of several connected items that are processed as a single unit. For example, the discriminant, $b^2 - 4ac$, consists of 6 items but it can be remembered and treated as a single chunk after much practice.

When a procedure consisting of many steps overloads working memory, write them down on the whiteboard as an external memory aid until it has been learned as a chunk. At this stage, various memory devices can be used to lighten memory load during processing, so that these devices become memory triggers at the retrieval stage later on.
Empowering Mathematics Learners through Effective Memory Strategies

Textual and pictorial memory devices are common in mathematics. Examples of textual memory triggers include:

- Mnemonics, e.g., “Please Excuse My Dear Aunt Sally” for PEDMAS, used to remember the order of arithmetic operations.
- Acronyms, e.g., SAS for Side-Angle-Side as a test of triangle congruence; TOACAHSOH in trigonometry.
- Songs, e.g., http://mathstory.com/index.html

Mathematical diagrams are powerful memory devices to help students process complex relationships, especially when they can convey the key ideas underlying mathematical patterns and proofs (Nelsen, 1993). This is illustrated by the two examples shown in Figure 3. In Figure 3(a), alternate angles are associated with a Z reminder, and Figure 3(b) gives a pictorial representation of the rule for the sum of consecutive natural numbers, \[ 1 + 2 + 3 + 4 = \frac{1}{2}(4)(1 + 4). \]

For some students, learning these memory triggers is as difficult as learning the actual contents. For instance, the mnemonic, BODMAS, does not indicate that division (D) and multiplication (M) belong to one group, and addition (A) and subtraction (S) to another group. Due to this ambiguity, students follow the order DMAS from left to right and make many mistakes. Teachers need to be extra cautious when they use such mnemonics in their lessons. Indeed, to be really helpful, these memory triggers should be accurate and easier to recall than the intended mathematical statements or formulae.

Another effective strategy to reduce cognitive load on working memory during problem solving is to exploit the goal-free effect (Sweller,
van Merrienboer, & Paas, 1998). Provide students with a typical problem but remove the question to be solved. Ask them to write down as many “answers” as they can generate from the given conditions. Consider the typical word problem below, where the question has been removed:

<table>
<thead>
<tr>
<th>Categories</th>
<th>Weekdays</th>
<th>Weekends</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regular</td>
<td>$12.00</td>
<td>$15.00</td>
</tr>
<tr>
<td>Student</td>
<td>$6.00</td>
<td>$8.00</td>
</tr>
<tr>
<td>Senior citizen (65 years old and above)</td>
<td>$8.00</td>
<td>$10.00</td>
</tr>
</tbody>
</table>

Mr Chan is 70 years old and he wants to bring his three school-going grandchildren to the movie.

The students may begin by working out the fare for different categories of audiences, but eventually, they will arrive at the intended question: the amount Mr Chan has to pay for four of them. Research shows that this goal-free process is less demanding on working memory compared to the means-ends analysis used to solve the intended question, thereby allowing the students to use their working memory more efficiently. Furthermore, this approach also promotes flexible thinking because students can pose different types of problems for one another to solve. This approach works well for problems that contain data and conditions, such as real-life contexts.

5.3 Pay attention to what is being processed

Another crucial consideration is what information students are engaged in during information processing. Teacher-designed worksheets often require students to fill in the blanks with key words, instead of writing out the complete statements. Consider the typical example below.
A percentage is expressed as a number out of ___________.
The percent symbol ( % ) is used to show that the number is out of _________.

The students just write down 100, %, 100; they do not process the complete sentences. With this curtailed form of information processing, they are unlikely to remember the complete statements, and this leads to poor understanding and recall. Getting students to write down complete statements or draw diagrams manually will take more time than filling in the blanks, but this generative learning process requires students to engage with the information at a deeper level, and this facilitates subsequent retrieval. Teachers also need to check that the students have copied the correct information into their notebooks.

6 Stage 3: Encode Information into Long-Term Memory

Newly processed information in the working memory has to be encoded into long-term memory for retention and future learning. Long-term memory is considered to have unlimited memory storage, and memory traces of information will last “permanently” in terms of days, months, and years. This reflects “permanent” changes in behaviours, thinking, emotions, and beliefs, which constitute real learning. The information can be encoded into three types of long-term memory: episodic memory, semantic (declarative) memory, and procedural memory.

6.1 Three types of memory in long-term memory

Episodic memory refers to memory of past learning and lived experiences. Students often remember their positive and negative experiences of learning mathematics more than the details of what they have learned. Positive experiences release dopamine, serotonin, and adrenaline, whereas anxiety and fear release cortisol, the stress hormone. According to Medina (2008), negative memories are more strongly encoded and persist much longer in LTM than positive or neutral ones. Based on Losada’s
recommendation (cited in Goleman, 2013, p. 173), mathematics lessons spread over several months should provide students with a 3:1 ratio of positive to negative experiences. This combination is likely to match with the “slightly stressful” option in emotion gateway discussed in Section 5.1. Teachers may also strive for a similar ratio in their teaching experiences. Hopefully, the 25% of negative lessons they may experience from deliberate attempts to hone new teaching skills are balanced by the 75% of positive lessons after they manage to fluently handle these new teaching techniques.

Semantic memory refers to memory of facts, rules, and statements; for example, 2 is a prime number but 4 is not. Verbalisers tend to encode these facts in word form whereas visualizers focus on imagery. For example, to compare the size of two unit fractions, a verbaliser may recall the statement, “for unit fractions, fractions with larger denominators are smaller,” whereas a visualizer may form mental pictures of the same circle being cut into different numbers of equal sectors and comparing the fractions by the size of the sectors. Students may have different preferences, but as noted in Sections 4 and 5.2 above, they must master both types of encoding.

Procedural memory, also called skill memory, refers to the memory of doing things physically, such as playing the piano, riding a bike, and running a 100-m race. These activities are part of the psycho-motor skills under education taxonomies. For mathematics, procedural memory includes application of steps for solving standard problems, construction of geometry objects using mathematics tools, plotting or sketching graphs manually, and the kinaesthetic actions undertaken to learn mathematics, such as paper folding, handling concrete objects, and measuring volumes of containers. By engaging repeatedly with these procedures, students form memories of how certain actions are carried out, in contrast to the semantic memory consisting of rules in words or images.

The CPA (concrete → pictorial → abstract) approach helps to build these types of memory: Concrete (procedural memory) → Pictorial (visual semantic memory) → Abstract (verbal semantic memory). Lessons that engage students emotionally (episodic memory) in CPA learning experiences are more effective than through only verbal explanations in
helping students to encode mathematics contents into LTM. Two major strategies to effect this encoding are discussed below.

6.2 Strategies to encode information into long-term memory

The first major strategy is for students to have the intention to remember, as discussed under Section 3. Students, who have already interacted with the new information via working memory but believe that they can search for it if needed later on, do not consciously aim to remember it and the information will not stick. It is important to include active memorisation of important points as a strategic learning objective for every lesson, even though education taxonomies (Anderson & Krathwohl, 2001) often treat memorisation as a low order objective.

The second major strategy to encode information into LTM is based on consistent findings in cognitive psychology, namely to build numerous meaningful connections of the new information with other types of information already in one’s cognitive structure. Three techniques for establishing conceptual connections are described below.

The first technique is to rehearse the information immediately, such as to repeat it verbally (auditory processing) and to write it down a few times (kinaesthetic processing) from memory. Engaging in such “maintenance rehearsal” seems to have disappeared from many mathematics lessons nowadays, even though it has short-term positive effects. Rehearsal can be spaced out by asking students to repeat these activities in different lessons.

The second technique is called “compare and contrast,” but it might be more effective to reverse the order and call it “contrast and compare”: first ask students to contrast differences and then to compare similarities. Mathematical items often look very similar with subtle differences that many students fail to pay attention to. Beginning with similarities could lead some of them to an illusion that they already know the ideas but actually have only superficial understanding. On the other hand, examining differences encourages them to pay close attention to subtle details. For example, students are often confused between these formulae: 
$$2\pi r, \pi r^2, \frac{4}{3} \pi r^3.$$ The contrast and compare technique works as follows:
Contrast differences in the powers of the radius ($r$); relate these powers to the idea of length (linear, $r$), area (square, $r^2$), and volume (cube, $r^3$).

Contrast differences in the coefficients; relate these values to the procedural memories of how they are obtained using the CPA approach.

Compare similarities, e.g., the formulae are about circular objects (circle and sphere), the presence of $\pi$, and multiplication.

The third technique to help students establish connections is to use mathematical diagrams and graphic organisers. These tools build on the “pictorial superiority effect”, a belief that concepts are easier to recall when they are learned through imagery than through texts. The “spatial contiguity effect” in cognitive load theory (Mayer & Moreno, 2003; Sweller, van Merrienboer, & Paas, 1998) suggests that text and diagrams should be placed close together to effect better retention and transfer and to overcome the split-attention effect. For example, statements about the three different types of roots of quadratic equations can be placed on top of the respective graphical representations, instead of given in the text.

Different types of graphic organisers, such as concept maps, mind-maps, KWL chart (What I already know; What I want to learn; What I learned), and Venn diagrams, can be used by students to connect mathematical ideas during learning and problem solving (Afamasaga-Fuata’I, 2009; Jin & Wong, 2011). The multi-modal thinkboard extends CPA to six modes of representation (Wong, 2015). These organisers should be completed by the students themselves as the lessons for a specific topic develop over weeks rather than to be given to them as completed maps to be memorised. These maps can be displayed as posters in the classrooms to stimulate informal, serendipitous learning. Students can use them as powerful memory triggers to aid retrieval.

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1 http://www.coolinfographics.com/blog/2015/1/20/the-key-to-infographic-marketing-the-psychology-of-the-pictu.html
7 Stage 4: Strengthen Information in Long-Term Memory

Even though information in LTM is thought to be “permanent,” it has been found to decay with time (“forgetting”) or subject to retroactive interference when new information is processed, as stated in Section 1.

Memory decay leads to the pruning of unused neurons, and retroactive interference results in new neural connections that may distort initial connections. When an item becomes activated with other information, it has to be stored again. This is an active re-construction process (Lewis, 2013). These new connections are built in the brain in unpredictable ways, and teachers can encourage students to use all available resources to strengthen and enrich the original connections in LTM. Three strategies are proposed below.

7.1 Deliberate practice

The adage “practice makes perfect” applies only if the practice leads to correct understanding. What is likely to happen is that “practice makes permanent” (Jensen, 2012) in the sense that practice, positive or negative, serves to make certain memory connections more “permanent.” Thus, students must practise rules correctly with timely feedback from their teachers.

After initial learning, three types of practice can be used to consolidate mathematical skills. They are called massed (block) practice, spaced (distributed) practice, and interleaved (mixed) practice. Reviews of research have concluded that spaced practice or interleaved practice is more effective than massed practice in the long term, but massed practice has short-term advantage (Brown, Roediger, & McDaniel, 2014; Medina, 2008; Rohrer & Taylor, 2007; Willingham, 2014). Instead of comparing one type of practice against another, an integrated, deliberate approach is recommended here:

a) Massed practice; students practise intensively several problems in one session or as homework to develop the new skill at the basic level and to build confidence.
b) Spaced practice; they continue to solve problems at increasingly difficult levels spread over many sessions, with feedback in between practice sessions.

c) Interleaved practice; within the same practice session, they solve problems covering different topics in order to discriminate among problem types and to counter problem fixation that arises when they focus on only one problem type at a time.

d) They should overlearn the correct procedures so that their working memory has more capacity to deal with higher order activities and solve challenging problems.

7.2 Testing effect

Many studies have shown that regular testing is more effective than re-reading in maintaining memory of the same materials and leading to better performance. This is called the “testing effect.” Taking regular low-stakes or ungraded quizzes as formative assessment also helps to alleviate students’ anxiety about testing and enhances their confidence if these quizzes are at the appropriate difficulty levels for them. On the other hand, repeated tests as high-stakes summative assessment may “reinforce the low self-image of the lower achieving students” (Harlen & Crick, 2002, p. 4). Many students are not aware of the benefits of this testing effect and as a consequence, they rarely self-test as a form of revision (Karpicke, Butler, & Roediger, 2009).

Fast-paced short drills, lasting no more than 10 minutes, can be conducted at the beginning of every lesson or during transitions between different episodes of a lesson. This helps students gain fluency about standard facts and solve basic problems correctly at the expected speed. These items can be made into flash cards, and students should prepare their own sets to encourage ownership. Free online drills are now available (e.g., Kahoot!\(^8\)), and they provide summary of responses, which helps the teacher to identify which items have not yet been mastered and who require special remediation. Group competitions and games such as bingo or “snap it” add interest to the drills. For items that have been learned by

\(^8\) https://getkahoot.com/
most students, it is still necessary to drill them on these items but only after longer gaps before re-testing. This approach is to avoid the illusion of knowledge (Brown, Roediger, & McDaniel, 2014), especially for low achieving students who tend to overestimate their competence, and, hence, do not make the requisite effort to improve further. Another technique to leverage on the testing effect is to ask students to set questions for their peers to solve. However, it is important to establish a routine to check that both the setters and solvers have produced the correct solutions, to avoid the unfortunate situations where errors become “permanent” through unsupervised practice.

### 7.3 Primacy and recency effects

During learning, students tend to remember best the first items (primacy), second best the last items (recency), and worst those items in between (Sousa, 2015). To counter these effects, teachers are advised to break a long session into separate episodes of about 10 minutes each, with clearly defined primacy and recency sections.

Similar effects may apply to revision used to strengthen memory. Students tend to revise topics in the order they are initially taught, which brings about these effects. Materials in between the first and last topics in the teaching sequence may not be remembered well. To counter these effects and to build on the interleaving effect, teach students to revise the topics in different orders and to test themselves instead of just re-reading worked examples. In particular, they should work through problems that they have made mistakes earlier. Teachers can make this more effective by providing parallel questions as practice after they have explained the correct procedures. Without this follow-up activity, the teachers cannot be sure that the students have understood the corrections.

### 7.4 Spiral scheme of work

Teachers can include the three types of practice, regular quizzes, and interleaved revision into their annual schemes of work (SOW). I call this new type of SOW a “spiral SOW” because it aligns with Bruner’s notion...
Empowering Mathematics Learners

of spiral curriculum (Bruner, 1960). Table 2 contrast the spiral SOW with the traditional SOW used in Singapore schools.

Table 2

<table>
<thead>
<tr>
<th>Traditional SOW</th>
<th>Spiral SOW</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1 to 3; topic A</td>
<td>Week 1; all 3 topics; basic level</td>
</tr>
<tr>
<td>Week 4 to 6; topic B</td>
<td>Week 2 to 3; topic A (deeper); quiz</td>
</tr>
<tr>
<td>Week 7 to 9; topic C</td>
<td>Week 4 to 5; topic B (deeper); quiz</td>
</tr>
<tr>
<td>Week 10; revision, term exam</td>
<td>Week 6; revise topic A and B; quiz</td>
</tr>
<tr>
<td></td>
<td>Week 7 to 8; topic C (deeper); quiz</td>
</tr>
<tr>
<td></td>
<td>Week 9 to 10; revision, term exam</td>
</tr>
</tbody>
</table>

Suppose the SOW will cover three topics, A, B, and C within a term consisting of ten weeks. The traditional SOW, shown in the left-hand column of Table 2, covers each topic sequentially for three weeks each, with the last week for massed revision before the term examination. The underlying principle is similar to massed practice. Although intensive last-minute revision can be helpful for some students, its effect is short-lived and does not make effective use of memory to consolidate initial learning.

In contrast, the spiral SOW, shown in the right-hand column of Table 2, covers all the three topics in week 1 at the basic level, so that the students gain an overview of these topics. Each topic is then covered for two weeks each at deeper levels. This generates one form of spaced learning. Quizzes are used to simulate interleaved practice based on the testing effect. The amount of time is the same for both versions, but the spiral SOW allows students to re-visit each topic at a deeper level over the whole term. Teachers are encouraged to conduct their own inquiry to test this instructional plan for their own classes.

8 Stage 5: Retrieve Information from Long-Term Memory

Retrieval is the process of accessing information stored in LTM to answer questions or solve problems. Failure to retrieve the relevant information is
considered a symptom of forgetting. However, in mathematics, students are rarely asked to solve problems that require verbatim recall of practised problems. Application or transfer, however minor, is required. The following three steps are involved:

a) Recall the relevant information and place it in working memory.
b) Re-construct meanings.
c) Use metacognition and executive control to solve the problem.

Thus, failure to solve a new problem is due to complicated interaction of “poor memory” with poor problem solving processes.

Retrieval is an active, re-constructive process, more than direct recall of information as stored. The success of retrieval depends on how well the information has been processed initially, encoded into LTM, and strengthened there with practice. If these earlier stages have been well executed, the chances of successful retrieval and transfer will be greater.

Table 3 outlines several factors that affect successful retrieval. Strategies for each factor are also indicated, albeit only briefly because they have been discussed above. On the emotional front, strong anxiety and stress are known to inhibit memory retrieval, even though some stress may enhance memory at the learning phase, as noted above (Vogel & Schwabe, 2016).
Table 3
Factors affecting retrieval and strategies

<table>
<thead>
<tr>
<th>Factors</th>
<th>Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Emotional forgetting due to anxiety and stress</td>
<td>• Recall positive episodic memory</td>
</tr>
<tr>
<td></td>
<td>• Give time for students to settle down</td>
</tr>
<tr>
<td></td>
<td>• Teach mindfulness techniques</td>
</tr>
<tr>
<td>Weak links among information</td>
<td>• Strengthen links through deliberate practice</td>
</tr>
<tr>
<td>Narrow spread of activation, i.e., the extent to which retrieval of one idea leads to retrieval of other ideas</td>
<td>• Build meaningful connections using graphic organizers</td>
</tr>
<tr>
<td></td>
<td>• Use memory triggers</td>
</tr>
<tr>
<td>Slow recall</td>
<td>• Students become aware of their own speed; work to increase it</td>
</tr>
<tr>
<td>Test requirements, e.g., use of calculators, time tests, question formats</td>
<td>• Students are aware of these conditions and practise accordingly</td>
</tr>
<tr>
<td>Environments, e.g., noise, lighting</td>
<td>• Improve conditions</td>
</tr>
<tr>
<td></td>
<td>• Students practise under examination conditions</td>
</tr>
</tbody>
</table>

9 Meta-Memory: Empower Students in Memory Use

Meta-memory is one type of metacognition\(^9\), covering knowledge and beliefs about one’s memory capacity and ability to monitor one’s use of memory strategies during study. To be empowered to use these memory strategies, students should gain some knowledge of brain anatomy and brain development and implications of neurological changes on memory and learning. Some of these ideas are included in the “My Maths Memory Checklist” given in the Appendix. Posters about memory can be placed around the school to increase exposure about memory in learning; see Jensen (2009) for ideas about this type of posters.

\(^9\) https://en.wikipedia.org/wiki/Metamemory
Students must be able to manage successes and failures that they will encounter in their learning. In the following scenarios, let S denote success and F “failure” which includes frustrations, obstacles, and struggles.

- **F → F → F …** This constant failure will result in learned helplessness, anxiety, and negative responses to new learning. Teachers need to provide opportunities for success, however minor, especially at initial learning.

- **S → S → S …** This constant success appears to be desirable, but it may indicate low expectations. Students who rarely fail may develop fear of failure and give up when the work is perceived to be too challenging. When students experience success, they should be praised for their effort rather than intelligence for their performance in order to promote a growth mindset (Dweck, 2012).

- **S → F → F …** In this case, the students experience initial success when the work is at basic levels, but fail to master more difficult tasks. This leads to loss in confidence and the perception that they are not “talented” in mathematics.

- **S (level 1) → F (level 2) → S (level 2) → F (level 3) …** This cycle of successes and failures at increasingly demanding levels will lead to deep learning. Students must understand this pattern and persevere accordingly. Secondary school students may be aware of this; for example, Cushman (2010) interviewed some American teens about how they mastered skills outside school situations and cited the following: “You might be good at it when you first start off, but you still got to practice so you can get better … They need help to see their own progress, so that they don’t only see how bad they are doing” (pp. 4 – 5).

Students also need to maintain a healthy brain. This can be done through proper diets (the brain takes up about 25% of energy intake), regular exercises, and adequate sleep and rest. Sleep has three significant effects on memory: information that is not strongly encoded is removed during sleep, resulting in forgetting of information that may be “unimportant” (synaptic downscaling); information may be consolidated and freely associated leading to new learning and creative problem solving (replay); emotional events are likely to be rehearsed and retained during sleep.
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(Carey, 2014; Lewis, 2013; Medina, 2008). A recent local study found that lack of sleep among teens inhibited new learning\(^{10}\).

Many students need encouragement and scaffolding to learn how to apply these memory strategies regularly. Teachers need to find out the effectiveness of the scaffolding they provide to their students by conducting action research. Administer the “My Maths Memory” checklist\(^{11}\) as a pretest, help students to develop one new habit or belief about memory use for about one month, complete the checklist as a posttest, and analyse the data. Repeat this cycle over several months to help their students consolidate new memory habits and develop constructive beliefs about memory.

10 Conclusion

In this chapter, a 5-stage information processing model is proposed to summarise significant ideas and strategies of how to help students enhance use of memory to improve mathematics learning. Briefly, students must be able to access the intended information through several sensory modes, overcome emotional anxiety, process new information in working memory, generate the intention to remember important information in long-term memory, strengthen and enrich the connections among stored information, and retrieve relevant information required for solving new problems. They can harness their brain power through staying healthy, adopting a growth mindset, and regularly applying deliberate practice to master new contents. Teachers can guide their students to deploy these memory strategies in sync with other learning techniques to deal with the memory problems they claim to impede academic progress among some of their students. Finally, the true indication of empowerment is whether the students themselves are able and willing to apply effective memory strategies to improve cognitive performance with a healthy sense of

\(^{10}\) http://www.straitstimes.com/singapore/health/teens-suffer-when-they-give-sleep-a-rest-study-shows

\(^{11}\) The checklist has not been validated; it is offered here to stimulate teacher inquiry into memory use for mathematics instruction.
enjoyment and confidence. Teachers play a critical role in empowering their students to reach their full potential in mathematics learning, partly but critically through the memory route.

References


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Appendix

My Maths Memory Checklist

- This is not a test. Tick the box to show how much you agree or disagree with each item.
- There are no right or wrong answers. Please answer the items honestly.
- Your responses will be used by your maths teacher to help you improve your maths memory.

SA = Strong Agree; A = Agree; D = Disagree; SD = Strongly Disagree

<table>
<thead>
<tr>
<th>Items</th>
<th>SA</th>
<th>A</th>
<th>D</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 I can remember maths better if it is given in words and/or formula.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 My memory of maths is poor when the lessons are stressful.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 If I work hard, I can remember most of the things taught in maths lessons.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 I can remember maths better if it is given in diagrams.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 My brain can grow when I use it to solve maths problems.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 I have to use my memory or I will lose it.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 Drawing graphic organisers (e.g., concept maps) helps me to remember important maths results.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 It is useful to review corrections to mistakes I have made in the past.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 I prefer to solve maths problems when they are grouped by the same types rather than mixed together.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 I can remember new maths better if it is connected to what I already know.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Date: ______________________

My “best” score on the memory test at http://cognitivefun.net/test/7 was: _____
Chapter 8

Empower Primary School Pupils to Use Representations to Solve Process Problems

YE O Kai Kow Joseph

A particular mode of representation cannot embody an abstract mathematical idea completely, it is necessary to use more than one representation. There is a need for primary school pupils to have a greater opportunity to learn mathematics using more than one representations that suit them best. When pupils solve mathematics process problems, teachers should offer an effective and empowering approach to create representations that help pupils see and express meaningful connections and patterns. This chapter therefore reviews the importance of multiple representations, discusses research studies related to solutions representations, and deliberates three process problems and its solution representations. Three process problems with solutions representations are highlighted so that teachers might trial in their mathematics lessons to develop pupils’ flexible connections among the multiple modes of representations.

1 Introduction

Mr Tan, a mathematics teacher, presented the following process problem to his primary six pupils during a lesson:

“Alan and Ben spent $200 altogether. \( \frac{1}{4} \) of Alan’s spending is $23 more than \( \frac{1}{5} \) of Ben’s spending. How much money did Alan and Ben spend?”
He asked the pupils for ideas on how they could represent or show the situation. He wrote their ideas on the board. The pupils suggested to use fraction discs, draw a picture, use square paper, make a table, draw model, and explain in words. Mr Tan told the pupils to represent the problem in more than one ways and that one way needed to be a visual or physical representation. This short preview into Mr Tan's mathematics lesson shows an importance on the use of different representations. He encouraged the pupils to suggest different representations before they even commenced to solve the process problem, and then he proceeded to let the pupils to decide for themselves which representations they wanted to use to represent the context of the process problem. In the traditional classroom, mathematics teachers normally showed their pupils how they should represent the problem such as drawing some visual representation. Pupils’ success in working through the mathematical problems depends on their ability to competently solve the problem in various ways as well as the ability to go back and forth among several different representations of the same problem.

In recent years, the principle of multiple representations has attracted much attention among mathematics educators. The National Council of Teachers of Mathematics (NCTM, 2000) encourage teachers and students to use multiple representations during mathematics instruction. It states that all students should “create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems; and use representations to model and interpret physical, social, and mathematical phenomena” (NCTM, 2000, p. 67). The NCTM’s Standards (2000) also suggest that a representation is not only a product (a picture, a graph, a number, or a symbolic expression) but also a process, a vehicle for developing an understanding of a mathematical concept and communicating about mathematics. An instructional issue is how many modes of representations need to be taught at various levels of mathematics instruction.

The latest mathematics syllabus in Singapore, which was released in 2012 and was implemented in 2013, continued to maintain mathematical problem solving as the central focus. It also encourages student to use heuristics to tackle a problem when the solution to the problem is not obvious. The heuristics include using representations such as drawing a diagram, tabulating and model drawing (Ministry of Education, 2012).
Mathematics educators, including school teachers are now beginning to use multiple external representations to solve problems. To serve as a vehicle in learning and communication, however, mathematics teachers need to be mindful that a representation must be personally appropriate and meaningful to a pupil. It must be part of primary school pupils’ daily mathematical lessons that slowly grows and develops over time. The main aim of this chapter is to consider the different ways in which multiple external representations are used as a tool to empower primary school pupils to solve process problems. This chapter therefore reviews the importance of multiple representations and discusses a few research studies related to multiple representation. This chapter also describes three process problems and their solutions representations that can be integrated in the teaching and learning of mathematics at the primary level.

2 Review of Literature

This section explains the importance of multiple representations and research studies related to the use of multiple representations when solving problem.

2.1 Importance of multiple representations

The use of multiple representation is a key emphasis found throughout the mathematics curriculum and textbooks. As mathematics is “inherently representational in its intentions and methods” (Kaput, 1989, p. 169), the mathematical relationships, principles, and concepts can be communicated in multiple representations including visual representations (i.e., diagrams, model drawings, pictures, or graphs), verbal representations (written and spoken language), and symbolic representations (numbers, letters). More than five decades ago, Bruner (1964) offered three modes of representation, namely the enactive, iconic, and symbolic, to be presented to pupils when learning mathematical concepts. Although the enactive stage may appear first as children develop, it remains important throughout children’s development, rather than being replaced by iconic or symbolic modes. In other words, iconic and symbolic modes remain as alternative
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representations, but with one providing a bridge to the other. Three decades later, Lesh, Post and Behr (1987) indicated that a pupil who grasps a mathematical concept “can (1) recognize the idea embedded in a variety of qualitatively different representational systems, (2) flexibly manipulate the idea within given representational systems, and (3) accurately translate the idea from one system to another” (p. 36). Their representation includes these five modes: contextual, visual, verbal, physical, and symbolic (NCTM, 2014; adapted from Lesh, Post, & Behr, 1987). It is important that primary school pupils are able to move between these representations. To help primary school pupils to move between these representations, pupils could be taught how to use a think-board. Haylock (1984) showed how the use of a think-board that comprise four parts labelled as numerals and signs, pictures, real things, and stories. These terminologies are easier to understand so that this think-board can be used as a learning tool. Seeger (1998) also supported that the process of representation or representing involves identification, selection, and discussing a concept through a mean that is fundamentally connected and easily understood. Recently, Cleaves (2008) also further classified six categories of mathematical representations: numerical/tabular, pictorial, graphical, verbal, symbolic (equations or expressions), and physical/concrete.

Researchers and mathematics educators also believed that multiple representations have many benefits in mathematics learning and teaching. Firstly, Kirwan and Tobias (2014) contend that no single representation is superior to another in all circumstances because each has its right in emphasizing different mathematical attributes or relationships in context. Secondly, the ability to use multiple representations and translate among these models is an important process of enhancing pupils’ mathematical understanding (Fennell & Rowan, 2001; Goldin & Shteingold, 2001). Pupils find it difficult to identify the relations among different representations. Perhaps in the mathematics instructions, teachers need to help pupils to move between representations as it supports the development of deeper conceptual understanding. While using representations are key to pupil’s understanding of mathematics concepts and the relationship among them, it is necessary to mention that each mode of representation provides only limited information and “stresses some aspects and hides others” (Dreyfus & Eisenberg, 1996, p. 268). Thirdly, pupils’ mistakes could also reveal their
difficulty with a particular representation, but not necessarily a lack of conceptual understanding underlying the problem (Flevares & Perry, 2001). However, selecting suitable representations will be more challenging for novice problem solvers than expert problem solvers because when novice problem solvers are trying to solve the problem, they may not understand the feature of the problem (Chi, Feltovich, & Glaser, 1981). Undeniably one distinctive competence is the knowledge of what representations are suitable for what problems (Kozma & Russell, 1997). One fundamental unresolved concern is how explicitly these skills should be taught; with some researchers arguing that teaching is essential (McKendree, Small, Stenning, & Conlon, 2002).

Other researchers such as Brenner et al. (1997, p. 666-67) argued that "attempts to teach problem representation strategies for mathematical problem solving have focused on teaching students ways to translate the words of a problem into other modes of representation using diagrams, pictures, concrete objects, the problem solver’s own words, equations, number sentences, and verbal summaries." A mathematical problem usually can be represented or solved in many diverse ways. The ability to solve a problem in multiple ways not only portrays a successful problem solver but also a more flexible thinker (Cleaves, 2008). In fact, solving a problem using more than one technique in a class situation often leads to a richer and livelier classroom discourse. As a result, multiple representations seem to be the important gateway and processes of mathematical understanding and it is often stressed that the ability to deal with them flexibly is critical for successful mathematical problem solving (Acevedo Nistal, van Dooren, Clareboot, Elen & Verschaffel, 2009; Even, 1990).

There is a general consensus that pupils would benefit from concepts presented in a variety of forms, and that they should be encouraged to discover different parts of each concept and solve problems through multiple representations (Goldin & Shteingold, 2001; NCTM, 2000). This observation was supported by Lesh, Post, and Behr (1987) and one of the main conclusions from their study states that “good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems that they instinctively switch to the most convenient representation to emphasize at any given point in the solution process” (p. 38). In summary, developing the learners’ competencies in dealing flexibly with multiple
representations should be a main goal in the mathematics instruction (Graham, Pfannkuch & Thomas, 2009).

2.2 Research studies on multiple representation

Mathematics education researchers have incorporated numerous theoretical perspectives to study the role of representations in problem solving (Cifarelli, 1988; Goldin, 1998). Earlier research studies by Yerushalmy (1991) found that even after extensive experience with multi-representational learning experiences designed to teach understanding of functions, only 12% of students gave answers that involved both the numerical and visual representations. Most answers showed the use of one representation and a neglect of the other. Such research suggests that appreciating the links across multiple representations is not automatic.

In another study, Cai and Hwang (2002) investigated 98 US and 155 Chinese Grade 6 students’ generalization skills in solving two pattern-based problems – “Dot and Doorbell problems”. This study showed that the disparities appeared to be related to students’ use of differing strategies. Cai and Hwang also found the Chinese students tended to select abstract strategies and symbolic representations while US students favoured concrete strategies and drawing representations. This study further confirmed the earlier studies conducted by Cai (1995a, 2000) that US and Chinese Grade 6 students differed markedly in their use of solution strategies and representations. The earlier Cai studies showed that US students frequently used visual or pictorial representations while the Chinese students used symbolic representations more frequently.

In another attempt to explore the connections between problem solving and visual abilities, Stylianou and Silver (2004) compared expert mathematicians with undergraduate students in an analysis of the potential and actual use of visual representations in mathematical problem solving. They found that although both groups acknowledged visual representations as worthwhile strategies, the experts used such strategies to a wider variety of problems (that included non-geometric examples). Further, the experts used the visual representations more regularly and dynamically to examine and understand the problem and plan a solution during the problem-solving process. Such studies indicate that the ability to develop and use visual
One of the most famous references in the literature regarding the use of visual representations in the work of mathematicians was probably made by George Polya (1945). Polya (1945) as well as Schoenfeld (1985) contended that visual representations such as pictures and diagrams are critical components in problem solving. Hence, it is worthwhile to deliberate the use of visual representations in the problem-solving process. Visual forms of representation have been perhaps the most researched topic among various representations in recent years because they are more readily available and they play a significant role in influencing and nurturing problem-solving ability. In problem solving, visual forms of representation can represent the structure of a problem. The correctness of a diagram for the solution of a problem depends on how well it represents that problem’s structure. Thus, it can be a useful tool in the solution of the problem. Polya’s reference to figures did not concern only the topic of geometry. On the contrary, he stressed that “even if your problem is not a problem of geometry, you may try to draw-a-figure. To find a lucid geometric representation for your non-geometric problem could be an important step toward the solution” (p. 108).

Larkin and Simon (1987) suggested that visual representations help the problem solver draw upon relevant preexisting knowledge and that visual representation use may facilitate the process of drawing inferences from new information in problem-solving situations. This was evident in lower secondary students in Singapore as they were still very comfortable using the model approach to solve problems (Fong, 1994) even though the Singapore’s Ministry of Education (2001) strongly encouraged secondary school students to use “varied strategies to solve problems” (p. 16).

Meanwhile, Wong (1999) also stressed that drawing diagrams is a significant problem-solving heuristic and many mathematicians employ visual imagery when they tackle problems. The mathematics education community has also been supporting the use of visual representations; recent reform efforts at all levels of mathematics education encourage the use of visual representations in mathematical problem solving. Research suggests that the use of visual representations can facilitate problem solving and

representational forms is a worthwhile problem-solving process that it should become an essential part of mathematical learning even at the primary level.
offered assistance at all phases of the problem-solving process. This was supported by Yancey, Thompson, and Yancey (1989) and they recommended that children must learn to draw diagrams. Stylianou (2002) also believed that mathematics students need training to acquire both the knowledge base and the strategies such as drawing diagrams to enable them to solve problems. However, it was found that students avoided using this mode to solve problems even when specifically told to do so (Vinner, 1989).

### 3 Solution Representations for Process Problems

There is a general agreement that the problem solvers’ representations play a central role in their problem solving. In solving a problem, a pupil first needs to formulate a representation of the problem based on his interpretation of the problem situations and conditions. From the representation, the pupil identifies the goals of the problem. After the pupil solves a mathematical problem, he may then use a certain representation to express solution processes in order to communicate the thinking involved in creating the solution. Thus, solution representations are the observable records produced by a pupil to communicate his thinking of the solution processes to a mathematical problem. In this chapter, solution representations refer to both the pupil’s thinking tools and his communication modes for solving process problems. Process problems are tasks that require pupils to develop general strategies for understanding, planning, and solving problems as well as evaluating possible solutions. In this sense, solution representations can be considered as a related, perhaps well-structured, version of solution strategies the pupil used in the solution process. The key idea of solution representations is to convey to teachers and primary school pupils that if they represent a problem in solution representations in several ways, their understanding of the problem will be enhanced. Therefore, solution representation is very closely related to problem-solving strategies. The following section describes three process problems with their solution representations that can be integrated in the teaching and learning of mathematics at the primary level.
3.1 Visual, verbal and algebraic representation

Consider the following process problem, shown in Figure 1, involving whole numbers that could be solved at the Primary 4 level.

**Process Problem 1: Whole Numbers (Primary 4)**
At a bakery, Jane paid $24 for 3 muffins and 4 cupcakes. Mary paid $18 for 3 muffins and 2 cupcakes. What is the total cost of one muffin and one cupcake?

**Figure 1. Process Problem 1**

Figure 2 is a visual representation for process problem 1. It shows that there are varied ways of expressing information and conditions and it is important to be able to go back and forth between representations.

<table>
<thead>
<tr>
<th>Represent a muffin by □ and a cupcake by ○.</th>
</tr>
</thead>
<tbody>
<tr>
<td>□□□□□ □□□□□ cost $24  and □□□□□ □□□□□ cost $18</td>
</tr>
<tr>
<td>Together, we have:</td>
</tr>
<tr>
<td>□□□□□□□□□□□ cost $24 + $18 = $42</td>
</tr>
<tr>
<td>So □□□□□□□□□□□ cost $42 ÷ 6 = $7</td>
</tr>
<tr>
<td>Therefore, the total cost of a muffin and a cupcake is $7</td>
</tr>
</tbody>
</table>

**Figure 2. Visual Representation for Process Problem 1**
It is given that:
3 muffins and 4 cupcakes cost $24 and
3 muffins and 2 cupcakes cost $18.
Together, we have 6 muffins and 6 cupcakes cost $42
So, 1 muffin and 1 cupcake cost $42 ÷ 6 = $7
Therefore, the total cost of a muffin and a cupcake is $7

Figure 3. Verbal Representation for Process Problem 1

Let m be the cost of one muffin and c be the cost of one cupcake
3m + 4c = 24 ---------(1)
3m + 2c = 18 ---------(2)
Adding Equation 1 and Equation 2, we have
6m + 6c = 42
6(m + c) = 42
m + c = 7
Therefore, the total cost of a muffin and a cupcake is $7

Figure 4. Algebraic Representation for Process Problem 1

Process problem 1 is typically solved at the secondary school level using algebra. When using algebra only, the problem involves forming two linear equations: 3m + 4c = 24 and 3m + 2c = 18. Figure 2, Figure 3 and Figure 4 show three different representations: visual, verbal and algebraic representation. In Figure 2, the total cost of one muffin and one cupcake is not apparent. One way to solve the problem is to combine the two situations so that we could get 6 equal groups of one muffin and one cupcake and then calculate the total cost of one muffin and one cupcake. One of the important factor to bear in mind is that as pupils’ progress through upper primary level, they are moving from the concrete operational to the formal operational
stage (in terms of Piaget's developmental stages). Thus, it is a right time to initiate pupils into visual forms of representing mathematical ideas and the notion that a pictorial representation can represent objects and relationships between the objects. The solution representations show that the visual representation provides a bridge to a verbal representation, which in turn provides a bridge to an algebraic representation.

3.2 Visual and numerical representation

The next level of problem solving requires accurate and perhaps even creative representation which can be shown by the following average problem in Figure 5. Two approaches of learning the concept of average are usually taught: an ‘evening-out’ approach and an algorithm (add-then-divide) approach.

**Process Problem 2: Average (Primary 5)**

Susan saved an average of $6.00 each day from Monday to Friday. On Saturday and Sunday she saved $9.50 each day. What was the average amount she saved from Monday to Sunday?

*Figure 5. Process Problem 2*

Cai (1995b) found that there are many sixth grade students knew the ‘add-then-divided’ algorithm for calculating average but only about half of the students showed evidence of having an understanding of the concept of average. If pupils have acquired sufficient skills in using representations, they will be able to solve the average problem using visual representation. Figure 6 is a visual representation for process problem 2. The solution representation in Figure 6 uses squares and rectangles to model the process of finding average and the second part of diagram shows the ‘evening-out’ process to help pupils discover the algorithm for finding average. Figure 7 shows the numerical representation where the average formula is used to solve the problem. As the two solution representations for process problem 2 show very different approaches to the problem, it is crucial to show
multiple representations. The evening out strategy in Figure 6 does actually also show that the total has to be 49, split evenly between the 7 days.

**Before**

![Image of square boxes with different amounts of money]

When we transfer $7 to even out the amount of saving on each square box, we are finding the average amount Susan saved from Monday to Sunday.

**Figure 6. Visual Representation for Process Problem 2**

Total spent during the week = $6 \times 5 \text{ days} = $30

Total spent during weekend = $9.50 \times 2 = $19

Average is $\frac{(30 + 19)}{7} = $7

Susan spends an average of $7 per day.

**Figure 7. Numerical Representation for Process Problem 2**

### 3.3 Pictorial, model drawing and algebraic representation

As primary school pupils are still at the concrete-operational stage and have problems conceptualizing mathematical abstractions, it will be more appropriate to use visual representation to illustrate the solution representation for process problem 3, shown in Figure 8.
Process Problem 3: Fraction (Primary 6)

Alan and Ben spent $200 altogether. $\frac{1}{4}$ of Alan’s spending is $23$ more than $\frac{1}{5}$ of Ben’s spending. How much money did Alan and Ben spend?

Figure 8. Process Problem 3

Figure 9 and Figure 10 shows the pictorial representations that could assist pupils to draw upon relevant pre-existing knowledge. The use of visual representation may facilitate the process of drawing inferences from the new information in the problem. The solution representation in Figure 9 and Figure 10 is intended to help pupils visualize abstract mathematical relationships and the accompanying problem structures. Primary school pupils will also view the process problem 3 as being more concrete and explicit and less abstract.

Since $\frac{1}{4}$ of Alan’s spending is $23$ more than $\frac{1}{5}$ of Ben’s spending, the five $\Delta$ can stand for the amount spends by Ben.

Ben’s spending: $\Delta$ $\Delta$ $\Delta$ $\Delta$ $\Delta$

Alan’s spending: $\Delta + 23$

$\Delta + 23$

$\Delta + 23$

$\Delta + 23$

$9\Delta = 200 - 23 \times 4 = $108

$\Delta = $108 \div 9 = $12

Alan’s spending = $(12 + 23) \times 4 = $140

Ben’s spending = $12 \times 5 = $60

Figure 9. Pictorial Representation for Process Problem 3
As a modeling approach permeates the school mathematics curriculum in Singapore, drawing bar diagrams will assist them to visualize the abstractions inherent in the problem (see Figure 10). Pupils can begin to use the bar diagram by drawing two rectangles. The first rectangle represents Alan’s spending and would be longer than the second rectangle that represents Ben’s spending. Basically, the bar diagrams uses rectangles to represent the quantities in a problem and shows the relationships among these quantities. This exposure to a less abstract representation may prepare pupils to work with letters as variables at primary 6 level; hence the pictorial yet concrete representation could serve as a bridge between working with numbers and the very abstract nature of letter-symbolic algebra.

![Figure 10. Model Drawing for Process Problem 3](image-url)

From this perspective, we are able to see that each unshaded bar is of equal size and could be represented as one unit while each shaded bar represents $23.

\[
9 \text{ units} = $200 - 23 \times 4 \\
= $108 \\
1 \text{ unit} = $108 \div 9 = $12
\]

Alan spends $12 \times 4 + 23 \times 4 = $140 while Ben spends $12 \times 5 = $60.
Primary 6 pupils may apply the standard algebraic strategy by transforming the process problem statements into two algebraic equations and solve them simultaneously (See Figure 11). However, primary 6 pupils may solve the problem by forming algebraic equations, they may be simply applying procedures that they have been taught without having a totally understanding the concepts behinds the procedures.

Let \( a \) and \( b \) be Alan’s and Ben’s spendings respectively

Then

\[
\begin{align*}
    a + b &= 200 \\
    \frac{1}{4}a &= \frac{1}{5}b + 23
\end{align*}
\]

Solving for the values of \( a \) and \( b \), the answers obtained are \( a = 140 \) and \( b = 60 \)

Alan spends $140 while Ben spends $60.

The three solution representations (see Figures 9, 10 and 11) illustrate how a link could be established. In fact, the model drawing and the algebraic representation in Figure 10 and Figure 11 have some similar features. In order to empower pupils to see the connection, a combination of the two approaches should be used to empower pupils to transit from pictorial to abstract.

These three process problems exemplify how process problems empower primary school pupils to explore various solution representations. However, there are potential limitations of pictorial representations in these three process problems. Pictorial representations require the numerical values to be positive integers. When negative integers and fractions are used in the computations, algebraic representations are more powerful and generalizable solution representations. The different representation that were discussed highlighted the different learning experience that pupils will gain when they work on process problems. This is only possible when the process problems that teachers use in their classrooms go beyond computations and rote algorithms. In addition, the ongoing use of multiple
representation will provide opportunities and possibilities for pupils to enhance their mathematical learning and understanding.

4 Concluding Remarks

Generally, primary school pupils need explicit instruction about representations before they are able to benefit from the cognitive advantages of representation use in problem solving. However, once pupils have developed an understanding of the structural relationships that can be represented by visual representations (i.e., diagrams, model drawings, pictures, or graphs), verbal representations (written and spoken language), and symbolic representations (numbers, letters), they become empowered as problem solvers because they can check the relationships among the information on a new problem against the types of relationships inherent in these multiple representations. This approach is predominantly useful for pupils who have difficulty recognizing the structure of a problem or are easily confused by surface details. Moreover, visual representations are not only a problem strategy in themselves but also act as a bridge towards algebraic representations. Thus, representations are important not only in the realm of skills but in enhancing mathematical concepts and problem-solving ability. For this reason, the use of multiple representations in mathematical learning, the connection, the coordination, and comparison with each other and the conversion from one mode of representation to another should not be left to chance, but should be taught thoroughly at the primary level, so that the pupils develop the skills of representing and handling flexibly mathematical knowledge in various forms.
References


Cifarelli, V. V. (1988). *The role of abstraction as a learning process in mathematical problem solving*. Unpublished doctoral dissertation, Purdue University, Indiana, USA.


Chapter 9

Empowering Mathematics Learners with Metacognitive Strategies in Problem Solving

LOH Mei Yoke      LEE Ngan Hoe

‘How do you tell the difference between metacognition and cognition?’ is a question often asked by teachers. The line of demarcation between them is often difficult to identify since the interactions between various mental processes are complex. As suggested by Flavell it is likely that metacognitive knowledge, metacognitive experience, and cognitive behaviour constantly inform and elicit one another when students work on a cognitive task. This chapter illustrates with examples how to identify notions of cognition or metacognition in students’ work in mathematics classrooms. Different types of students’ activities may lead students to exhibit a range of cognitive and metacognitive strategies in mathematical problem solving. Such insights help teachers to better structure their instructions to address both cognitive and metacognitive strategies – an essential ingredient in empowering students as self-directed learners.

1 Introduction

The Singapore Mathematics Curriculum is guided by the pentagonal framework (Figure 1) and central to this framework is mathematical problem solving which is supported by five inter-related components: Concepts, Skills, Processes, Metacognition and Attitudes.
While this problem solving framework has been in place since 1990s (MOE, 1992), only more recently, Metacognition has been identified as one of the four competencies of the domain ‘Critical and Inventive Thinking’ in the emerging 21st century competencies framework (MOE, 2014). It is described as follows:

Metacognition refers to thinking about one’s own thinking – that is, gaining an awareness of and control over one’s own thinking through reflection to become a more effective thinker and learner. (MOE, 2014, p. 7)

Metacognition in mathematical problem solving can improve students’ mathematics performance (Mevarech & Amrany, 2008; Schenfeld, 1992) as well as prepare them for future learning and the future workforce. However, although metacognition has been the emphasis in both the Singapore Mathematics Curriculum and 21st Century competencies, very little is known how metacognition has worked in the Singapore mathematics classroom. Teachers, in general, may not be
familiar with teaching and assessment of metacognitive strategies in mathematical problem solving. They were often confused over cognition and metacognition (Lee, Ng, Seto & Loh, 2016). In order to incorporate metacognitive strategies into their instruction to build and enhance problem solving competency as well as 21st century competencies, teachers need an in-depth understanding of both cognition and metacognition and how to differentiate between cognitive strategies and metacognitive strategies.

2 Definitions of Metacognition

Flavell (1976) who first coined the term ‘metacognition’ defined it as follows:

Metacognition’ refers to one’s knowledge concerning one’s own cognitive processes and products or anything related to them... metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or objective. (Flavell, 1976, p.232)

Flavell’s definition of metacognition refers to knowledge about and of cognition, and processes of monitoring and regulating of cognitive actions. Flavell’s work provided the foundational knowledge for the theory of metacognition. The notion of metacognition was further developed by numerous researchers who were interested in the psychology of metacognitive thinking (Brown, 1987; Efklides, 2001; Garofalo and Lester, 1985; Lester, 1994; Schoenfeld, 1987; Schraw & Moshman, 1995). In recent years, many researchers seem to adopt a two-domain description of metacognition: knowledge of cognition and regulation of cognition (Nietfeld, Cao & Osborne, 2005; Pintrich, 2002; Schraw & Dennison, 1994; Schraw & Moshman, 1995). Knowledge of cognition involves awareness of and knowledge about one’s own cognition and task requirement; regulation of cognition involves activities that help control one’s thinking or learning which includes planning, monitoring and
evaluation (Jacobs & Paris, 1987; Pintrich, 2002; Schraw & Moshman, 1995).

In the local context, the Singapore Mathematics Curriculum defines metacognition this way:

Metacognition, or ‘thinking about thinking’, refers to the awareness of, and the ability to control one’s thinking processes, in particular the selection and use of problem-solving strategies. It includes monitoring of one’s own thinking, and self-regulation of learning. (MOE, 2012, p.19)

This definition seems to encompass the metacognition aspects similar to most researchers. ‘Awareness of one’s thinking processes’ in the description above seems to be similar to research that defines such awareness as knowledge of cognition (Nietfeld et al, 2005; Pintrich, 2002; Schraw & Dennison, 1994; Schraw & Moshman, 1995). The ‘ability to control one’s thinking processes’ which includes aspects of monitoring and self-regulation is similar to the definition of regulation of cognition by Lucangeli and Cabrele (2006). The curriculum document also details some tasks that help to develop students’ metacognition. These include problem solving tasks that require planning, evaluation, use of thinking skills or heuristics; monitoring and regulating tasks such as the ‘thinking aloud’ method to describe strategies used in solving problems; group discussion, deriving alternative solutions to a problem and checking reasonableness of the answer (MOE, 2006). These suggested tasks involve metacognitive processes in relation to problem solving situations.

In this chapter, metacognition in mathematical problem solving is operationalized as consisting of three interdependent components: metacognitive knowledge, metacognitive monitoring and metacognitive regulation. This definition is closely aligned to the definition of metacognition as described in the Singapore Mathematics Curriculum (MOE, 2012). Though literature review (Nietfeld et al, 2005; Pintrich, 2002; Schraw & Dennison, 1994; Schraw & Moshman, 1995) has shown that it is difficult to classify observed behaviours under monitoring and regulation separately, classifying the observed behaviours more finely
would gain insights into metacognitive processes employed during mathematical problem solving.

The three components are described as follows:

_Metacognitive Knowledge_ refers to an individual's awareness of his or her own cognitive and affective resources (Chang & Ang, 1999) in relation to the task. Metacognitive knowledge is composed of three types of awareness, namely declarative (knowing what), procedural (knowing how), and conditional (knowing when and why) (Brown, 1987; Hacker, 1998; Schraw & Moshman, 1995). From this perspective, students devise or select strategies to solve a mathematics problem based on their metacognitive knowledge about the problem and the strategies or heuristics in problem solving.

_Metacognitive Monitoring_ refers to periodic engagement in understanding the task performance while executing the cognitive actions (Schraw & Moshman, 1995). It includes actions that keep track of problem solving activities throughout the phases of problem solving.

_Metacognitive Regulation_ refers to decisions made after re-evaluation of cognitive and metacognitive activities throughout the problem solving process (Brown, 1987; Efklides, 2001; Pintrich, Wolters & Baxter, 2000). Metacognitive regulation is a more observable aspect of metacognition when it results in decision made to existing plans, and/or monitoring actions, that lead to change in strategic actions based on existing knowledge. However, regulation may also result in non-actions e.g. student progresses with plan after checking on workings half-way through executing the plan.

3 Difference between Cognition and Metacognition

Flavell made a distinction between cognitive strategies and metacognitive strategies. While a cognitive strategy is ‘to help you achieve the goal of whatever cognitive enterprise [domain-specific knowledge] you are
engaged in’, a metacognitive strategy is ‘to provide you with information about the enterprise [domain-specific knowledge] or your progress in it’ (Flavell, 1985, p.106). In reality, a combination of the different elements of knowledge is at work at the same time. Schoenfeld (1987) categorized problem solving behaviours into two types: tactical and managerial. He defined tactical behaviours as the ‘things to implement’ such as the use of algorithms or heuristics in solving a problem. This seems to describe the cognitive elements of problem solving. Managerial behaviours include episodes and executive decision points primarily for the purpose of analysing problem solving moves. This seems to relate to metacognitive decisions.

4 The Mathematics Problem

In this chapter, we define a mathematics problem as a mathematics question which a problem solver does not have a direct or immediate path to a solution. This has a similar definition in NCTM (2000) that states

“problem solving means engaging in a task for which the solution method is not known in advance. This would mean that a student is only considered engaged in mathematical problem solving when the student is attempting a mathematics problem. In order to find a solution, students must draw on their knowledge” (p. 52).

5 Methodology

The analysis of students’ work in this chapter is part of a bigger study which has a sample size of 783 Secondary One students. The students were each given a mathematics problem to solve. Besides giving the solution, the students were required to report retrospectively the problem solving process. This allowed processes that were not reflected in the working to be captured. The mathematics problem is as follows:
Three students, Ali, Sam and Don were given the following problem:

- 2A4, 329 and 5B3 are 3-digit numbers.
- When 2A4 is added to 329, it gives 5B3.
- 5B3 is divisible by 3.
- What is the largest possible value of A?

Ali thought A could be 1.
Sam thought A was 5.
Don thought A was 4.
Who was correct?

This question required students to assess reasonableness of the solution against stated conditions. There were about 5 conditions, some were stated clearly in the question e.g. 5B3 is divisible by 3, largest possible value of A while others were implicit e.g. the value of B is 2+A+1 where the 1 ten comes from adding 4+9, and that 2+A+1 cannot be more than 9 since there is no renaming to hundreds place. In order to solve the problem successfully, students needed to monitor their steps to ensure all the conditions were fulfilled.

6 Analysis of Students’ Work

Examples of students’ work and retrospective self-reports would provide insights of students’ understanding of the cognitive and metacognitive strategies employed in solving the problem. Figure 2 shows Student G’s solution and retrospective self-report.
From the working, only procedural steps were shown and solution seemed to suggest that Student G tried to substitute A with 1, 5 and 4. If the sum 5B3 is divisible by 3, that would be a possible answer. How the student decided on the method and the process of thinking were not reflected in the working. There was no telling if the ‘x’ (cross) was marked while doing the first step or after all the three calculations were done. On the other hand, the retrospective self-report gave a glimpse into the thinking involved. While the choice of method was not explained, Student G clearly had a plan to try ‘all the numbers’ for A. Student G thought the answer was five initially (A = 5) as it was the largest given A. Student G monitored the calculation while executing his/her plan, ensuring that the condition of divisibility by 3 was met. Realizing that the division resulted in a decimal (194.333), Student G regulated his/her action by deciding on the alternative possible solutions, A = 1 and 4. Since 4 is larger than 1, Student G indicated that Don (A=4) was the answer. It showed once again that Student G monitored the solution to ensure all conditions stated in the question were met.

The method of systematic substitution is a cognitive strategy. The systematic procedural steps and calculations (cognitive aspects) showed...
Student G had good metacognitive knowledge of this cognitive strategy. The decision in using this particular method was metacognitive in nature. Student G exercised metacognitive knowledge, metacognitive monitoring and metacognitive regulation during the execution of the plan to ensure that he/she was on the right track.

Figure 3 shows another example of a student’s solution and retrospective self-report. In this case, Student H employed a different cognitive strategy from Student G. Student H has good metacognitive knowledge of how and when to execute the method as shown in the systematic way in executing the steps and clear explanation of the method described in the retrospective self-report. Initially, Student H wrote ‘first I
add numbers from 214, 224, 234, … 264 until it isn’t 500 anymore’. It showed that Student H’s intention was to use numbers up to 264 but as he/she monitored his/her working, he/she realized that the total would still be below 600 and not ‘isn’t 500 anymore’. Student H then regulated his/her actions by taking one more step, add 274 to 329 that resulted in a sum above 600, to ensure that he/she had exhausted all possible number such that the sum was 5B3. Student H further reported that he/she monitored the execution of the method and ‘looked at those which I ticked and look what I have added….’ before he continued to fulfil the condition stated in the problem i.e. the largest possible value of A. This is, in fact, an example of non-action for a regulation activity as mentioned in the definition of metacognitive regulation.

In the above examples, it showed that most of the working or calculations could only show the cognitive strategies or a sense of the metacognitive knowledge of the cognitive strategies. Retrospective self-report provided information that was not revealed in the working especially the metacognitive monitoring and regulation aspects.

Interactions with teachers seemed to point towards a misconception held by teachers that metacognitive monitoring is associated to the execution phase of problem solving only while metacognitive regulation only occurs after execution of the plan to solve a problem. The next two examples would show how the metacognitive monitoring and regulation take place throughout the problem solving process. Figure 4 and Figure 5 are examples of metacognitive strategies employed in metacognitive monitoring and regulation respectively.
Figure 4 shows Student J recalled key points (‘wrote down notes’) by carefully attending to the information in the problem at the beginning of the problem-solving process and then ‘labeled and organized’ the working
to ensure that he/she could ‘complete the question without mistakes’. These were aspects of metacognitive monitoring throughout the entire problem solving process. While knowing the cognitive strategy of systematic substitution is important, employing metacognitive monitoring strategies helps in deriving an accurate solution.

Figure 5. Student J’s retrospective self-report

Figure 5 shows how Student J did not stop even though he/she thought that an answer was found. He/She regulated his/her action by continuing with the ‘third number’. Metacognitive regulation occurred throughout the entire problem solving process.

The examples thus far showed that students have good knowledge of the cognitive strategies. They knew the procedural steps well. The next example in Figure 6 showed that having the knowledge of the cognitive strategy but lack metacognitive strategies might not lead to a correct solution.
The boxed-up portion in Figure 6 showed that Student M did not actively monitor the execution of the plan. He/she did not employ metacognitive monitoring strategies such as reading the question again to ensure that the conditions stated were met. In this case, ‘7’ was not one of the three given numbers for A. The lack of metacognitive monitoring and regulation strategies had led to a wrong conclusion.

7 Conclusion

In this chapter, we have shown that both cognition and metacognition are important for successful problem solving. However, teaching the cognitive strategies may not be sufficient in empowering our students in problem solving. Students need to develop good metacognitive knowledge of the cognitive strategies. Some suggestions for teachers to consider in developing students metacognitively in problem solving:
a. To activate students metacognitively, begin by guiding students in understanding the problem. This includes understanding the mathematics language in the problem and using strategies such as highlighting important information, drawing appropriate representation of the information provided or recalling key points through writing short notes. Research has shown that metacognitive students were better in organizing and processing information, they tended to compare different strategies and use different representations while deciding on the appropriate strategy to solve a problem (Mevarech & Amrany, 2008).

b. Get students to articulate their thoughts through writing retrospectively on their problem solving process periodically or interview students on their problem solving process. These tasks not only make students’ thinking visible to the teachers, they also stimulate metacognition e.g. self-checking (Clarke, 1992). While such tasks take up time in class, students being their prime witnesses to their own thinking, there is no better way to find out about their thinking (Solas, 1992).

c. Use problem solving frameworks such as The Problem Wheel (Lee, 2008), Pólya’s four phases of mathematical problem solving (Pólya, 1957) that have an emphasis in self-regulation or self-questioning to guide students through the problem solving process. These frameworks guide students in exercising metacognitive knowledge, metacognitive monitoring and metacognitive regulation throughout the entire problem solving process.

As metacognition does not develop automatically in all students (De Jager, Jansen & Reezigt, 2005), a more metacognitive-oriented learning environment is believed to empower our students in mathematical problem solving.
Acknowledgement

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References


Empowering Mathematics Learners


Chapter 10

Mathematical Problem Solving: An Approach to Empowering Students in the Mathematics Classroom

TOH Tin Lam

In line with the move towards a student-centric classroom and away from the traditional teacher-centric one, it is understandable that a greater sense of student empowerment should be realized in the classroom. From studies in Self-Determination Theory, learner empowerment is closely correlated to their motivation, which in turn is an important part of the learning process. This chapter demonstrates how the entire mathematical problem solving process is closely aligned to realizing student empowerment in the mathematics classroom. The well-known Polya’s four-phase problem solving model will be used as the framework for our discussion. A mathematical task the “Flower Bed Problem”, which has been discussed elsewhere, is used to demonstrate how student empowerment and problem solving are closely related. Also student empowerment can be realized in the daily mathematics classroom with appropriate re-crafting of some existing mathematical tasks and a re-conceptualization of the lesson implementation using the problem-solving framework. While students are given autonomy in choice-making in their own learning journey, reflective processes should be embedded to harness the benefit of student empowerment.
1 Introduction

Educators in the west have generally agreed that “empowerment” in the education context seems to originate from educators’ belief that “[n]egotiations around power are an unavoidable part of the teaching-learning process” (Tai, 1998, p. 426).

This chapter begins with an exploration of the notion of “empowerment” in the context of secondary school mathematics education and, in particular, “in-the-classroom empowerment” of which teachers would have an important role to play. Usually, empowerment in education is associated with tertiary or adult education, during which an individual can make choices with their own learning (Brooks & Young, 2011). For students at the primary and secondary education, most activities are deemed to be mandatory; the list of compulsory items includes the common syllabus which is rarely student-controlled.

In the present day secondary school education scene, which focuses on student-centred instruction, is there any room for “student empowerment”? We will show that in fact, a natural progression of mathematical problem solving, which is the heart of the mathematics curriculum in many countries of the world, is student empowerment. Several examples related to this will be discussed.

2 Notion of Empowerment

Empowerment is a multidimensional social learning process that helps people gain control over their own lives (Page & Czuba, 1999). Empowerment can be manifested at least in two levels: collective or individual (Hur, 2006).

Individual empowerment involves how people think about themselves and the knowledge, capacities, skills, and mastery they actually possess (Staples, 1990, p. 32). Hur (2006) further identified four dimensions of individual empowerment: (1) a sense of meaning, (2) competence, (3) self-determination, and (4) impact.

The general objective of empowerment is to engage an individual with tasks that are meaningful to him or her. Out of a sense of meaning of the task an individual identifies, competency is generated in the individual.
Empowering and Problem Solving

He or she will then be able to develop a state of understanding of what to do in a novel situation or in resolving a particular problem (self-determination). This in turn results in an impact towards the organization and society. One can easily appreciate the relevance of the objective of individual empowerment in the school context and, in particular, in the mathematics classroom. Most mathematics teachers and educators would undoubtedly agree that the key objective of mathematics education is that an individual can perform problem solving, that is, to be able to handle a novel situation for which the solution might not be immediately forthcoming for the individual.

Let us consider a typical mathematics classroom. Ideally, teachers should develop in their students a sense of meaning in what they learn. With this as the base, students are led to build their competency in the tasks, which will lead them to understand how they should function in a novel situation. This description is a reminiscent of what mathematics teachers are familiar with – mathematical problem solving, which is the heart of our Singapore mathematics curriculum since the 1990s. Thus, it will appear that the truest spirit of mathematical problem solving could be closely linked to student empowerment. The next section presents some background discussion of mathematical problem solving from the education literature and in Singapore context.

3 Problem Solving

3.1 Mathematical problem solving in the curriculum

Despite the many regular mathematics curriculum reviews in the Singapore Ministry of Education (MOE), mathematical problem solving as represented by the “pentagonal framework” has been the heart of the Singapore mathematics curriculum since the 1990s (Ministry of Education, 2006; 2013). Much of the theory of problem solving has been established since Polya’s first book on problem solving How to Solve It (Polya, 1945). Now what awaits the educators is to follow through the “hard and unglamorous work” in practical terms (Schoenfeld, 2007).
3.2 Problem solving model

Presumably every good mathematics student would have invented his or her own model of problem solving, a blueprint that an individual problem solver would follow in the event of handling an unseen problem whose answer or solution is not immediately forthcoming to the solver. Here we select Polya’s four-phase problem solving model as our framework since this model is relatively well-known to most teachers, and that it is the model used in the Singapore mathematics curriculum document (Ministry of Education, 2006, 2013). Toh, Quek, Leong, Dindyal and Tay (2011) adopted the model shown in Figure 1.

![Figure 1. Polya’s model of problem solving](image)

Polya’s model begins with understanding a given problem, and the strategies that the individual uses as attempts to overcome difficulty in understanding the problem, based on contextual cues and other available resources such that the problem solver’s prior knowledge, interest etc.

The second stage of this model, namely, “Devise a Plan”, describes the problem solver’s plans to tackle the problem. In the Singapore syllabus document, this consists of a list of “heuristics” – which are rules of thumbs which one could apply when one is stuck at a particular problem – that an individual can attempt in situations when one is stuck at a particular problem and presumably has understood the problem.
The third stage, namely, “Carry out the Plan”, involves the actual process of the problem solver going through the process of solving the problem. Should one encounters difficulty in completing the solution, one is left with the choices of (1) persisting with the original plan but needs to examine the given conditions more carefully; or (2) abandon the plan and go through the stages of Devising the Plan or Understanding the Problem again. The decision making process here is correlated to an individual’s metacognitive ability; students should be given opportunity to participate more extensively in this process, and the solver’s decision-making process is an important part of his or her learning.

A problem solver needs to have the opportunity to utilize his or her resource, make decisions on how these resources can be utilized at his or her disposal. This efficiency of an individual’s decision making in utilizing the resources is known as “control” or “metacognition” in Schoenfeld’s language (Schoenfeld, 1985).

The fourth stage is “Check and Extend”. Note that in Polya’s original words, this last stage is known as “Look Back”. In Toh et al (2008, 2011), this modification was made, as it was felt that this stage has the great potential for an individual learner to not just check the correctness or reasonableness of one’s solution, but to be more forward looking, by, for example, considering alternative solutions and compare their relative merits, to extend and generalize the given problem. The last part of extending and generalizing a given problem has the potential to allow the students to stretch their potential to the fullest and which might lead to advanced mathematical thinking and even mathematical discovery. Much of this gives students the autonomy to propose their own problems and stretch themselves to the furthest that they choose for themselves.

Anecdotal evidence shows that the teaching of problem solving in Singapore schools has usually emphasized the teaching of heuristics. The correct choice and use of heuristics was thought to be the most sufficient facet of successful problem solving (Toh, Quek & Tay, 2008).

With regard to attempts to teach problem solving, Schoenfeld (1985) suggested that students should be equipped with more than just Polya’s model and a range of heuristics. The students must have the opportunity to manage resources at the disposal, choose promising heuristics to try, control the problem solving process and their own progress. It is implied
in this suggestion that most of the decision making processes in problem solving must be returned to the learners instead of decided by the teachers; this resounds much with empowering students in their learning processes.

By empowering the students through problem solving, based on the understanding of this problem solving model, students will be encouraged to move beyond the mere completion of the solution to a particular problem, but also to stretch the problem in depth and breadth depending on the students’ inclination and capacity. We could see glimpses of student empowerment in mathematics classroom if this entire process of problem solving is embedded into the regular mathematics curriculum.

Experience usually tells us that some problems are crafted in such a way that are more naturally inclined for the problem solvers to explore several heuristics, as these problems might appear to be more inviting to different approaches, while other problems are more “instructional” and which problem solvers merely need to follow through a series of guided procedures. Other problems with fixed answers and exact number of numerical values required to solve the problem are more prone to lead students to follow fixed procedures with little room for decision making.

### 3.3 Mathematics questions commonly encountered by students

As discussed in the preceding subsection, the entire problem solving process, when enacted in full, naturally leads to student empowerment in allowing them to make decisions on how far they would want to progress on the journey of solving a particular problem. However, this would mean that the curriculum material and the types of questions must be suitably crafted to meet the need to allow students greater autonomy.

One particular observation made by Fan and Zhu (2007) on their study of the Singapore mathematics textbooks shows that problem solving heuristics is introduced separately from, rather than infusing into, the main content topics. In addition, this entire business of problem solving, which is the core of the Singapore mathematics curriculum, was treated as one independent topic and appeared in the form of drills. They also found that significant discrepancy existed between the curriculum document standards and the textbook developers’ interpretation and implementation
of problem solving.

In addition, the textbooks are usually replete with questions with very tidy answers. Even though there are sections on “real-life applications” of the particular mathematical concepts, the very tidy state of scenarios with perfect final answers appears to be specially crafted for the students. These artificially created problems can hardly convince students of real-life applications of mathematics (Ang, 2009), what more about autonomy or giving students the control in their own journey of learning mathematics.

A scan through past year examination questions from the National examinations and school examination papers shows that most of the assessment problems in mathematics that students are generally exposed to are highly structured; there is hardly any room for students to make decision on the appropriate strategies that they might be able to apply. This is understandable under high-stake examination condition with tight time constraint; students are assessed on their ability to apply particular knowledge and skills, rather than on their selection of appropriate strategies or the exercise of controls in problem solving. If one believes that assessment drives the way the curriculum is delivered, it might not be surprising that students might not be sufficiently exposed to mathematical tasks that they are given much autonomy or opportunity to make much decision in mathematics classrooms on problem solving tasks.

4 Mathematics Tasks for Empowering Learners

4.1 The circular flower bed task

We begin with an example of a task that can be used in a mathematics classroom to engage students in a problem solving task in which they have the autonomy to make decisions, exercise their judgement in completing the task. The task is shown in Figure 2. Kaur and Toh (2011) discussed in detail how this task was used to engage student teachers in problem solving in the teacher education programme in Singapore. It is also a good problem to introduce in the secondary school mathematics classroom as well.
A CIRCULAR FLOWER BED

A landscape gardener uses exactly 36 paving bricks 22 cm by 11 cm to form a “circular” flower bed. However, the gardener does not want any spaces between each brick, so she cuts the bricks so that the face of each brick is in the shape of an isosceles trapezium.

Determine the shape of the required brick. Make a model from cardboard. Give details of the steps taken. State any assumptions that you have made. Clearly show all mathematical calculations.


Figure 2. The Flower Bed Task

This Flower Bed Task could be used as an activity to reinforce students’ familiarity with Polya’s problem solving model (understand the problem, devise a plan, carry out the plan and look back). It could also reinforce students’ problem solving habits in class while giving them the full autonomy of constructing the “circular flower bed” subject to the constraints of the question. To begin with, it might be a challenge to understand this problem as in what it means by a “circular” flower bed. The list of scaffolding questions discussed in Kaur and Toh (2011) could be adapted for use in the secondary school classroom context.

A particularly interesting feature of the Flower Bed Task lies in its possibility for infinitely many solutions (although two of the most common solutions which do not involve the slicing of the bricks into more layers are discussed in Kaur and Toh (2011)). While this task provides the opportunity for students to be involved in the actual work of constructing the flower bed, it could also urge them to discover the mathematical knowledge that is needed to solve the task (trigonometric ratios in triangles). The learners could have the choice of deciding the mathematical
content knowledge that they need in order to provide an exact solution for the task.

A reflective learner should ideally come to realize that indeed there are infinitely many solutions to this task (if one were to slice each brick into a fixed number of layers). The two solutions provided in Kaur and Toh (2011) do not involve slicing of bricks. In these two solutions, the bricks are cut into isosceles trapeziums to be put together into a “circular” structure without gaps. The cut can either be on the longer sides or the shorter sides, thereby giving two distinct solutions. Consequently, the size of the circular flower bed could be arbitrarily large. Through having the opportunity to make decisions on various sensible solutions, students are engaged in deep discussion on the mathematics behind the task, and the learners might even decide to acquire the more profound concept of limits and infinity.

In carrying out this task, the role of the classroom teacher lies in facilitating students to go through the various stages of mathematical problem solving, rather than providing the students with the “correct” solution or teaching them the methods that could lead to a correct solution. In empowering the students, teachers could see that their role has evolved from being an instructor to a facilitator and advisor; they could advise the students on the relevant content knowledge (or “cognitive resource” in Schoenfeld’s language) needed to solve this particular problem; the responsibility and decision lies in the students.

It is crucial that students be given time to consider all the various aspects of the problem, and teachers provide appropriate scaffolding as and when necessary. A thorough discussion of the appropriate scaffolding and different levels of scaffolding within the framework of Polya’s problem solving model can be found in Toh et al (2011, pp 21 - 23).

Students should be given the opportunity to conceptualize the entire problem solving process with the teacher functioning as facilitator instead of “teacher”. Even towards the solution, teachers should allow, and even encourage, students to use different approaches to arrive at their proposed solution, within the conditions of the given problem. At the stage of the students arriving at their answer, the teacher could trigger the students to think of other possible solutions, and how to “improve” their solution if possible. A vision of this problem solving class is one in which students
are actively engaged in a mathematics discourse through lively discussion and, most importantly, students are the leaders of the discussion, with teachers at the background providing the facilitation for the discourse. This is distinctly different from the traditional teacher-centric lessons in which the teacher is delivering the lessons; here, students decide on their solution and, most of the time, lead the discussion.

### 4.2 The percentage task

The preceding subsection shows an example of a class activity in which students are empowered with making decisions in a mathematical problem solving task. Any classroom teacher would appreciate that most day-to-day classroom lessons are not of the above structure as teachers need to cover core content within a very tight schedule. In fact, research has shown that time pressure plays a significant role in a teacher’s pedagogical decision (for example, Black 2004). Black (2004) found that a lack of time forced the teachers to cover the content quickly, thereby the depth of the mathematical discourse might be compromised.

Are there other opportunities of “empowering” students in the usual mathematics classroom, besides implementing relatively “large scale” tasks such as the Flower Bed Task described in the preceding section? Here, we shall discuss other “smaller scale” empowerment activities in the usual mathematics classroom, using appropriate mathematical tasks. We shall demonstrate how standard textbook mathematical tasks, appropriate for grade 7, can be adapted or constructed for the purpose. The same approach is suitable for all other levels.

Consider a typical task, T1, involving lower secondary mathematics topic on percentage.

**T1.** Joseph bought a shirt at $50. He was at first given a 10% discount. The shopkeeper decided to give him a further 10% discount. Find the total price that he paid for the shirt.

The objective of T1 is to engage students to compute percentage discounts using their knowledge of percentage, and trace the sequence of the two consecutive events of percentage discounts. This is a slightly more
challenging task as it involves multiple steps. This task addresses some of the common misconceptions students may develop about “addition of percentage” (Lee, 2009).

However, T1 provides little room for the students to be stretched beyond having their possible misconceptions addressed, since (1) all the essential numbers for computation are given; and (2) the attention of the task is directed at the final answer (the price that Joseph has to pay).

To give students greater room to manipulate the task and to make decision in the problem solving process, one could consider modifying T1 by: (1) removing some figures for computation (in this case, the amount $50); and (2) modifying the task to one which focuses not on the final answer, but on the impact of the two processes. The modified task becomes:

T2. Joseph bought a shirt. He was at first given a 10% discount and then given a further 10% discount. What is the total percentage discount?

The modified T2 appears to be lacking of information on first sight. This will be useful to throw the decision to the students on how to proceed: Use algebra? Substitute appropriate figures and then look for pattern? Or to discard the task as it might be faulty?

Some students might solve this task by drawing model diagrams; other students might use the heuristics of substituting numbers (the cost of a shirt) and obtain an answer. They might see that the solution is not perfect, as what is true for a particular number might not be generalizable, and thereby yearn for a “generalizable” method. The classroom teacher should optimize students’ learning experience through such a task, facilitate them to progress on their plans using their approaches, and eventually lead them to appreciate the underlying intent of the task (that the result is not 20% in this task).

To reiterate the point, the important decision on the approach to solve the task should be left to the students: the teacher’s role lies in not providing the solution, but providing appropriate scaffold using the
problem solving framework, in order to develop the habit of the mind to think along problem solving.

Teachers should further use this opportunity to reinforce in their students the habit to check the correctness or reasonableness of their solution, or the elegance of the solution that they provide. For example, if students substitute numbers to obtain their answer in T2, they could be led to see that this is at most a partial solution, as it is not immediately obvious that the final answer will be unchanged should the value was different from the substituted. Moreover, such a solution done by substituting specific values for the unknown is not elegant (although it is a legitimate heuristic approach); for students who draw models to obtain the answer for T2 could realize that this approach might not be generalizable if the given conditions of percentage in T2 are replaced by other numbers. It is crucial that students have the ownership of examining their own solution, and have the enthusiasm to explore how the given problem could be further generalized. Such a passion should be kindled by the classroom teacher facilitator within such a classroom setting. The extent of generalization of the problem is an important decision that students should be empowered to make.

It must be emphasized at this juncture that in order that students be empowered to make decisions at the various stages of solving such problems, they must have been exposed to and familiar with the entire problem solving processes. Importantly, teachers should have the tolerance to allow their students to struggle in solving mathematical tasks, especially when they are not able to obtain the solutions immediately; teachers should only provide scaffolding as and when necessary.

A list of other similar problems that have been modified from typical school textbook questions and which could be used to engage students in such higher order thinking skills and which empower them in the problem solving processes is given in Figure 3.
1. If the area of a circle is increased by 100%, what is the corresponding increase in its radius in percentage?

2. If the perimeter of a square is increased by 50%, what is the percentage increase in its area?

3. Sam went to a shopping centre. He was given a discount of 10% and followed by another discount of 10%. What is the total percentage discount that he enjoys?

If a shop decides to charge 7% GST followed by 10% service charge and another shop decides to charge 10% service charge and followed by 7% GST, will there be any difference to the consumer? What about the government (who collects the GST)?

4. If the area of a circle is increased by 100%, what is the corresponding increase in its radius in percentage?

2. If the perimeter of a square is increased by 50%, what is the percentage increase in its area?

3. Sam went to a shopping centre. He was given a discount of 10% and followed by another discount of 10%. What is the total percentage discount that he enjoys?

If a shop decides to charge 7% GST followed by 10% service charge and another shop decides to charge 10% service charge and followed by 7% GST, will there be any difference to the consumer? What about the government (who collects the GST)?

4.3 The average speed task

We next consider another task on average speed that is in the Lower Secondary Mathematics curriculum. Both classroom anecdotes and literature contains evidence that students have difficulties and misconception about average speed (see for example, Reed & Jazo, 2002). A commonly discussed misconception is that the average speed of two parts of a journey is misinterpreted as the average of the two speeds is prevalent among secondary school students. Thus, the following type of questions might be common in the lower secondary school mathematics textbooks:

T3. John travels from town A to town B which are 100 km apart with an average speed of 40 km/h. Immediately he returns
from B to A with an average speed of 60 km/h. Calculate his average speed for the whole journey.

In solving T3, and using the correct “resource” (in our case, the formula for average speed), it is not too difficult for students to discover that the average speed for the whole journey is not equivalent to the average of the two speeds. To give students greater autonomy to explore concepts related to average speed, and eventually to generalize the task, T3 can be modified to the following:

T4. A car travels from A to B along the straight line with a constant speed of \( u \) and immediately returns from B to A with a constant speed of \( v \). Which is larger: the average of the two speeds or the average speed of the whole journey?

Here the modification is similar to T1: (1) remove the numerical values from the original question; and (2) modify the question from one which focuses on computing an answer to one in which the solvers are required to analyse the attribute of the task structure.

Similar to T3, this task invites the solver to critically examine the concepts of “average speed of the whole journey” versus “average of the two speeds”. Furthermore, similar to T2 which appears to be lacking information at first sight, T4 gives the solver the autonomy to either solve the task heuristically or, for the more advanced solver, tackle the task algebraically with appropriate formulation (since numerical values for calculation are not provided).

By using the heuristic approach of substituting values (for \( u \), \( v \) and the distance between A and B), students might wonder whether the result (average speed is not equal to, and does not exceed, the average of the two speeds) holds true for all other values of \( u \) and \( v \). After a few attempts of substituting different values, it might appear that the average speed not exceeding the average of the two speeds might be true in most cases, and curiosity could lead them to explore the relation between them.

The more advanced solver might end up exploring concepts involving arithmetic mean (AM) and the geometric mean (GM), and ideally the AM-GM inequality. Compared to T3, T4 has more room for students to explore
and for students with different abilities to solve the task according to their level and interest. The teacher facilitator could even fire the passion in the higher achieving students to explore more tasks related to AM-GM-HM inequality, and examples of these in the real world.

The teacher should function as a facilitator who leads his or her students along their interest and ability, and fire the passion into the students to explore the same question within their ability and stretch beyond.

Tasks like T4 are not entirely difficult to craft. As discussed above, teachers can modify these tasks from existing textbook tasks and, more importantly, conceptualize the whole process of facilitation to empower students with decision making in problem solving. At this juncture, one other problem that might be of this nature is Question 7 of Toh et al (2011, p. 93). This problem is an adaptation of Terence Tao’s airport-surprise puzzle in https://terrytao.wordpress.com/2008/12/09/an-airport-inspired-puzzle/.

4.4 Student reflection

A class in which students are empowered is typically one in which students have the choice on the tasks that they wish to perform (subject to their inclination and interest), while teachers are able to meet the specific instructional objectives for the particular lessons. Instead of assigning a set of tasks in which students must complete all the questions, a list of activities (such as Figure 3) could be assigned to the students, who could make a decision on a stipulated number of tasks to complete. They are not required to solve all problems, but are allowed to select the problems that they are more naturally inclined or interested.

To further enable students to reflect on the choice of their tasks, students could be invited to communicate on the choice of the tasks. In particular, it is useful for students to reflect on why they have avoided particular tasks and chosen other tasks instead. The teacher’s role is to facilitate the students in solving the problems that they have selected through the problem solving process.

Students’ reflection is an important part of learning. Studies have shown that opportunities that encourage student reflection, instead of the
traditional teacher teach-explain structure, have resulted in greater student achievement (for example, Wheatley, 1992). To encourage reflection in such a problem-centred class, journal writing is one possible approach that can be carried. Frequently, students should be required to share the types of problems that they enjoy solving, and those that they found difficulty and also to share some parts of the problem solving processes that they would like to share. Student reflection after the problem solving process allows them to reflect on their choices that they have made that have made satisfactory progress and those that might have not been satisfactory; this is a suitable learning process which will not only allow students to take ownership of their reading, but will also deepen their understanding of mathematics and problem solving. An example of a reflection format based on the problem solving process is suggested in Figure 4.

**Reflection:**

Problem (please write the problem here)

1. One thing I find easy about solving this problem
2. One thing I find difficult in solving this problem
3. One important decision that I have made in solving this problem that is successful.
4. How I think this decision can be used in other problems / contexts
5. Two things to remember in solving this problem

*Figure 4. Reflection*

The reflection as suggested in Figure 4 is aligned to Stage Four (Looking Back Stage) of Polya’s problem solving model. It is crucial that students be given the opportunity to develop this reflection strategy as their problem solving habit. A natural progression of sustained Looking Back at the solution of a problem would eventually lead to students’ comparison
of various solutions, extending and generalizing the given problem, which is the habit of mathematicians (Toh et al, 2011).

5 Conclusion

This chapter discusses ways in which students could be given more opportunity in the mathematics classrooms to be involved in choice-making and greater autonomy. Such opportunities range from large-scale problem solving activities in which students would need to make sound decisions in completing their tasks (such as the Flower Bed Task in Figure 2) to smaller activities of adapting textbook problems to reinforce problem solving habits (Figure 3). The use of reflection component (e.g. Figure 4) could further reinforce such habits. This approach aims to develop in students the desirable habits in mathematical problem solving and provides an opportunity of student empowerment even in the secondary school mathematics classroom.

Research on motivational theory and self-determination theory have found that learner empowerment is an essential part of learner motivation, and hence integral to the process of learning (Houser & Frymier, 2009). It is usually seen that students’ autonomous choice-making, or self-determination, is a feature of tertiary and adult education and not a part of secondary education. However, the bridge between the secondary education in which individuals undergo mandatory learning process to this feature of adult education in which the learners have the freedom to choose is perhaps student empowerment.
References


Empowering and Problem Solving

Chapter 11

Empowering Mathematics Learners through Exploratory Tasks

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Mathematical empowerment is often connected to the ability to fluently use mathematical knowledge and skills in various contexts or situations. Consequently, mathematics learning should support students’ acquisition of not only concepts, but also skills and various strategies of problem solving. One of the possible ways to empower mathematics learners is through the use of exploratory mathematics tasks. An exploratory mathematics task could lead students to an exploratory activity, which includes interpreting situations in mathematical terms, formulating mathematics questions, making conjectures, exploring various strategies, and making generalizations. This chapter provides examples of the use of exploratory mathematics tasks in classroom practices. The tasks did not provide any indication about the required mathematic concepts. In their early work, many students used non-mathematical strategies. The students started exploring mathematics concepts after the teacher scaffold them to use some mathematics to solve the tasks. This finding suggests that an exploratory task is not a standalone feature to empower mathematics learners. Teachers’ scaffolding is an important requirement to guide students in dealing with exploratory tasks and in developing their ability to use mathematics.
1 Introduction

Empowering mathematics learners has long been considered as an essential goal of mathematics education. In the 1990s, Skovsmose (1994) promoted a so-called ‘critical mathematics education’ in which he highlighted that teachers should implement mathematics curriculum as an empowering activity for their students. A call for mathematical empowerment has also been approached at the policy level. Clements and Ellerton (1996) suggested that reforming mathematics education required collaborative work between curriculum developers and teachers to develop curriculum that is empowering both for teachers and their students. In recent years, empowering mathematics learners is often connected to mathematical literacy and considered as an important key not only for higher education, but also for economic purposes and full citizenship (Doyle, 2007; Stinson, 2004). This view emphasizes that mathematics should not be seen as an isolated concept that is apart from students’ life. Instead, mathematics should be seen and treated as an integral part of students’ life.

According to Ernest (2002), mathematical empowerment concerns knowledge and skill to use and apply mathematics. It deals with students’ acquisition of mathematical knowledge and skills, and also problem solving skills. Mathematically empowered students could “demonstrate an appropriate range of mathematical capabilities such as performing algorithms and procedures, computing solutions to exercises, solving problems” (Ernest, 2002, p. 2). Furthermore, these students also have a good ability in applying mathematics concepts, carrying out approaches in solving mathematical problems, and judging the correctness of solutions. Doyle (2007) noted that key to mathematical empowerment is mathematical literacy and that mathematics tasks play a crucial role. Stein, Smith, Henningsen & Silver (2009) too emphasized that cognitive demands of mathematics tasks determine the development of students’ mathematical thinking and skills. This chapter discusses the use of exploratory tasks to empower mathematics learners by considering the role of mathematics tasks for mathematical empowerment.
2 Exploratory Tasks

Exploratory tasks are tasks that “may lead students to exploratory activity, from which they do substantial work and learn new mathematics” (Ponte, Mata-Pereira, Henriques, & Quaresma, 2013, p. 11). Exploratory tasks are not only used to support the development of new concepts, but also to develop students’ ability to apply mathematics concepts they have learnt and to connect between concepts. Exploratory tasks are ill-structured and do not have straightforward solving strategies (Cifarelli & Cai, 2005). Another characteristic of exploratory tasks deals with the type of information available in the tasks. Exploratory tasks mostly have either superfluous information or incomplete information. Tasks with superfluous information mean that the tasks contain irrelevant data so that the solvers need to identify and select only the relevant data. Tasks with incomplete information indicate that some important data are missing in the task so that the solvers need to gather the required data either by estimation, through multistep procedures, or search from other sources.

To solve exploratory tasks students need to reformulate the problem statements, interpret the problem situations in terms of mathematics, and generate mathematical models. During the exploratory activity, students are actively involved in exploring strategies or relationships, mathematical reasoning, making conjectures, and communicating mathematical ideas, and interpreting and validating the reasonableness of mathematical results.

3.1 Designing exploratory tasks

Most tasks in Indonesian textbooks have low cognitive demands and focus on procedural knowledge (Wijaya, Van den Heuvel-Panhuizen, & Doorman, 2015). Figure 1 shows examples of tasks available in Indonesian mathematics textbooks. Task 20 mainly focuses on factual knowledge, i.e. about the definition of parts of a circle. This task does not offer enough opportunities for students to engage in a mathematical exploration. Task 4 requires students to prove a particular mathematical situation. Although this task is of a higher cognitive demand, the exploratory activity is limited to mathematical exploration because the task is situated in intra-mathematical context. In this case, students do not
get opportunity to do a wide range of exploratory activity which includes mathematical modeling.

### Figure 1. Examples of textbook tasks
(Note: translation of Problem 4 and Problem 20, As’ari et al., 2014, p. 81)

- 4. Look at the figure.
  - Line $k$ is the perpendicular bisector of chord $AB$.
  - Line $l$ is the perpendicular bisector of chord $CD$.
  - Point $P$ is the intersection of line $k$ and line $l$.
  - Does point $P$ lie on the center of the circle? Explain your reasoning.

- 20. Look at the figure. Based on the figure, give examples for the following parts of a circle:
  - a. Radius
  - b. Diameter
  - c. Chord
  - d. Sector
  - e. Arc
  - f. Segment
  - g. Apothem

By referring to the characteristics of exploratory tasks as mentioned earlier, a regular textbook task as shown in Figure 1 can be modified into an exploratory task. Figure 2 shows an example of an exploratory task. To facilitate students’ modeling process, this exploratory task is situated in an extra-mathematical context and cannot be solved by straightforward strategies. In the case of Figure 2, the context is about an archaeological artefact for which no clear indication about mathematical procedure is provided. Students need to transform this real-world problem into a mathematical problem. At this stage, students’ exploratory activity focuses on interpreting the mathematical meaning of “reconstructing a broken plate into its original size and form”. The exploratory activity continues when students work within the mathematical problem or model; for example exploring mathematical concepts or procedures which are relevant to constructing a circle from a given arc. An exploratory task
could also lead to hands-on activity or paper-and-pencil exploratory activity.

Figure 2. Broken Plate Task: An example of exploratory task (adapted from Kordemsky (1992))

3.2 Computer-based exploratory tasks

According to Clements (2000), computers offer students flexibility in exploring various possible strategies and solutions for mathematics problems. Considering this potential of computers, an exploratory task can also be developed in the form of a computer-based task. A computer program or software that can be used to make a computer-based exploratory task is GeoGebra. According to Dikovic (2009), GeoGebra is beneficial for mathematics learning because it: (1) has easy-to-use interface, (2) could encourage students to do mathematical projects, experimental and guided discovery learning, (3) stimulates students to personalize their creations through exploration of features, and (4) helps students gain a better understanding of mathematics concepts.

Figure 3 is the Broken Plate Task which is presented as a computer-based exploratory task using the GeoGebra software. Before students begin the exploratory activity using GeoGebra, they need to interpret the broken plate task as a mathematical problem. After the students have
recognized the mathematical aspect of the task they can use the features of GeoGebra to explore strategies for solving the mathematical problem.

3 Using Exploratory Tasks in Mathematics Classrooms

This section illustrates how the exploratory task, Broken Plate, was enacted. The two forms of Broken Plate Task were used with two different groups of first-year university students, enrolled in a teacher training program. One group worked with the paper-and-pencil version of the exploratory task (shown in Figure 2) while the other group worked with the GeoGebra version of the same task (shown in Figure 3). Although the participants of this study were university students, their prior geometrical knowledge was relatively similar to that of senior high school students as they had not yet learned college geometry.
3.1 *Paper-based exploratory task: Unsystematic exploration*

The paper-and-pencil version of the Broken Plate Task was given to 32 student teachers. They were given the task, a sheet of paper containing a large figure of the broken plate, blank papers, scissors, compass, and ruler. The student teachers worked in groups of four or five. Every group discussed the mathematical meaning of ‘reconstructing the broken plate into its original size and form’. All the groups understood what they had to do, i.e., construct a complete circle which is coincident with the broken plate. At this stage, however, none of them mentioned the centre of circle. They did not notice that to construct the circle they needed to determine the centre of the circle. This shows that the student teachers mainly focused on the shape of a circle and did not really consider the mathematical properties and construction of a circle.

However, in the next stage when they had to draw the circle, many of them still proceeded without drawing on any properties of circles, for example how to find the centre and radius, etc. Figures 4, 5 and 6 show some approaches adopted the students when attempting to construct the circle.

### Fold-cut

1. Fold the paper along the imaginary chord connecting the edges of the arc of the broken plate. In this case the imaginary chord becomes the line of symmetry.
2. Cut the paper along the arc of the broken plate.
3. Unfold the paper.

*Figure 4. The fold-cut approach*
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**Cut-rotate**
(1) Cut the figure of the broken plate
(2) Place it on a paper and trace the arc of the broken plate on the paper
(3) Rotate the broken plate such a way so that a part of its arc coincides with the arc on paper
(4) Trace again the arc of the broken plate on the paper
(5) Repeat step (3) and step (4) until a full circle is constructed

![Figure 5. The cut-rotate approach](image)

**Cut-duplicate-cut-tessellate**
(1) Cut the figure of the broken plate
(2) Make duplicates of the broken plate and cut them
(3) Tessellate the arc of the broken plate and its duplicates in such a way to construct a circle

![Figure 6. The cut-duplicate-cut-tessellate approach](image)
Only one group of students appeared to be engaged in a mathematical exploration activity. Although they still used a cutting strategy, as shown in Figure 7, they incorporated mathematical ideas in their strategy. After cutting the broken plate into half, they folded the plate symmetrically to get a line of symmetry. Next, they folded the broken plate on other part and obtained the second line of symmetry. They argued that both lines of symmetry must include diameters of the required circle and these diameters intersected at the centre of the circle. After they located the centre, they constructed a circle by using a compass.

**Half-cut – double fold – draw a circle**

1. Cut the broken plate
2. Fold it symmetrically to get a line of symmetry
3. Fold it symmetrically to get another line of symmetry
4. Construct a circle with the intersection of the lines of symmetry as the centre

*Figure 7. The half-cut-double fold-draw a circle approach*
3.2 GeoGebra-based exploratory task: A tendency towards feature exploration

Similar to their counterparts who worked on the paper-based exploratory task, 19 student teachers who worked on the GeoGebra-based exploratory tasks also understood that, mathematically, the task was about constructing a complete circle which was coincident with the broken plate. Between these two groups of participants, the exploratory activity in constructing a circle was rather different. Unlike the paper-based task which could lead to a non-mathematical exploration, the GeoGebra-based exploratory task was more inclined to engage the students in a mathematical exploration. This is so as GeoGebra supports a mathematical environment and the features of GeoGebra connect with mathematics concepts.

However, as shown in Figures 8, 9 and 10, during the exploratory activity the student teachers only focused on the features of GeoGebra and did not really consider mathematics concepts. For example, students who used the feature ‘Circle through three points’ could not give a mathematical explanation of why a circle can be constructed from three given points.

**Circle through three points**

1. Choose the feature ‘circle through three points’
2. Put three points along the arc of the broken plate
3. Construct a circle by using the feature

*Figure 8. Circle through three points*
**Exploratory Tasks**

**Semi circle – reflect**

(1) Choose the feature ‘Semicircle through 2 points’

(2) While the feature is activated, put a point on one edge of the arc of the broken plate and construct a semicircle in such a way it coincides the arc of the broken plate

(3) Draw a line connecting the two edges of the semicircle

(4) Choose the feature ‘Reflect object in a line’

(5) Click the semicircle and then the line until a circle is constructed

![Figure 9. Semi circle - reflect](image)

**Reflect**

(1) Draw a line connecting the two edges of the arc

(2) Choose the feature ‘Reflect object in a line’

(3) Click the broken plate and then the line

(4) The constructed figure is not a circle

![Figure 10. Reflect](image)
4 Teachers’ Scaffolding: From Unsystematic Exploration and Feature Exploration to Mathematical Exploration

The classroom activities show that both exploratory tasks have prompted student teachers to explore strategies, which later could develop students’ creativity. However, their explorations were mainly unsystematic and feature-oriented. Therefore, a teacher can play a crucial role in guiding the unsystematic and feature exploration towards mathematical exploration. According to Blum (2011), teaching tasks which require mathematical modeling should emphasize on guiding students to construct knowledge or identify mathematics concepts actively and independently by using their prior knowledge and experiences. A key aspect for such teaching is keeping a balance between providing guidance and fostering students’ independence. For this purpose, a teacher could use flexible interventions and metacognitive prompts to elicit students to reflect on their own understanding of the problem and on what basis they selected the mathematical procedures to solve the problem. Regarding the use of exploratory tasks in mathematics classrooms, teacher’s scaffold could be in the form of questioning. Questioning can be used: to focus thinking on mathematics concepts; to help students extend their thinking from concrete and factual knowledge to analytical and evaluative aspect; to help students see connections between different mathematics concepts or between mathematics and real-world contexts (Swan & Pead, 2008). This chapter illuminates two types of questioning, “can you” and “what if”, that were used to scaffold student teachers in exploration of the Broken Plate task.

4.1 “Can you” question

The first type of question that was used to guide student teachers in their mathematical exploration was “can you find another strategy?” It was observed that this question directed students to look for other strategies. However, their new strategies were still at the same level, either unsystematic or feature-oriented. For example, students who used ‘circle through 3 points’ came up with ‘semi-circle and reflect’ as a new strategy without giving any explanation related to mathematical properties of circles. It seems that the words ‘other strategies’ were not specific enough
to direct student teachers to think of the mathematical properties and construction of a circle. Considering this response, a more specific “can you” question was posed, e.g. “Can you do it without cutting the broken plate?” which was posed to groups of student teachers who worked with paper-based tasks. By limiting students’ strategy, which in this case was cutting, directed student teachers to explore other tools and strategies. A group of students who previously came up with the strategy ‘half-cut – double fold – draw a circle’ discussed how they could construct lines of symmetry without cutting and folding paper. These students finally noticed that the lines of symmetry could be constructed by drawing the medians of chords as shown in Figure 11 (Note: constructing apothems was also observed in the groups of student teachers who worked with GeoGebra-based task).

“Can you do it without cutting the broken plate?”

Figure 11. Can you do it without cutting the broken plate?

4.2 “What if ...” strategy

According to Kaur (2012), a “What if” task which included modified given information that could direct students to re-examine a task and see the
impact of the changes in the task on the solution process. Taking a similar perspective, a “what if” question was posed to direct students to mathematical exploration. To some student teachers who worked with GeoGebra-based exploratory task, the question “what would you do if we used a compass instead of GeoGebra?” was posed. The modified information in this question is GeoGebra, which is replaced by compass (note: the compass was imaginary because student teachers were still working with GeoGebra). After group discussion, student teachers who previously used ‘Circle through 3 points’ strategy noticed that three points on the arc of the broken circle made a triangle. As their new strategy, these students constructed a triangle whose vertices lay on the broken plate and then constructed a circumscribed circle of the triangle as shown in Figure 12. (Note: this circumscribed circle also emerged in the groups of students who worked with paper-based task after teacher scaffold their thinking with the same questioning strategy).

Figure 12. What would you do if we use a compass?
5 Concluding Remark

This chapter provides an example of an exploratory task that is modified from a textbook task. A textbook task often provides only information required to solve it and also gives a clear indication about the required mathematics concepts or procedures. Such tasks can be modified into exploratory tasks by reducing some information and making them ill-structured. Furthermore, an exploratory task can be presented as a paper-based task or a computer-based task. An exploratory task could lead students to unsystematic and non-mathematical explorations. Therefore, the effectiveness of an exploratory task to empower mathematics learners also depends on teachers’ scaffolds during students’ exploratory activity. Questioning is one means and it can guide students towards mathematical exploration while keeping them active and independent during the exploration. Specifically “Can you” and “What if …” questions were found to be effective in moving students from unsystematic and feature-oriented explorations to mathematical explorations.

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References


Chapter 12

Use of Open and Guided Investigative Tasks to Empower Mathematics Learners

Joseph B. W. YEO

In this chapter, I will discuss how to use open and guided mathematical investigative tasks to empower secondary school students to think and solve problems like mathematicians. Although the mathematical contents of these tasks are at the secondary school level, the thought processes are similar to what mathematicians do in their academic lives: posing problems to investigate, specialising (i.e. examining specific cases), conjecturing, justifying (i.e. testing and proving conjectures) and generalising. The reader may come to appreciate that these mathematical thinking processes are not very different from 21st century competencies needed to solve real-world problems in the workplace or in daily lives.

1 Introduction

The focus of the school mathematics curricula in many countries is on mathematical problem solving, e.g. in the United States of America (National Council of Teachers of Mathematics, 1980), in the United Kingdom (Cockcroft, 1982), in New Zealand (Ministry of Education of New Zealand, 2007) and in Singapore (Ministry of Education of Singapore, 1990).

Some of these curricula specifically include mathematical investigation as an important element of problem solving. For example, the key ideas of the Australian curriculum for mathematics state, “Students develop the ability to make choices, interpret, formulate, model and
investigate problem situations, and communicate solutions effectively” (Australian Curriculum, Assessment and Reporting Authority, 2016), and the Singapore school mathematics curriculum defines mathematical problems to include “a wide range of situations from routine mathematical problems to problems in unfamiliar context and open-ended investigations” (Ministry of Education of Singapore, 2000, p. 10).

But how is mathematical investigation different from mathematical problem solving, and why do we have to bring investigation into the mathematics classrooms? The next section will seek to answer these questions.

2 Mathematical Investigation

In this section, what constitutes a mathematical problem and a mathematical investigation will be discussed, followed by a deliberation on some similarities and differences between a mathematical problem and a mathematical investigation. The section will then end with some reasons for engaging students in mathematical investigation.

2.1 What is a mathematical problem?

In the literature, there are generally two views of what a mathematical problem is. The first view holds that whether a situation is a problem or not depends on the particular individual (e.g. Henderson & Pingry, 1953; Lester, 1980). If a student can solve a mathematical task easily, then the task is not a mathematical problem to him or her. The second view holds that whether a mathematical task is a problem or not depends on the nature and purpose of the task. For example, the purpose of most textbook tasks is to “provide students with practice in using standard mathematical procedures” (Lester, 1980, p. 31) while the purpose of problem-solving tasks is for students to make use of some problem-solving heuristics, such as looking for patterns, to solve it although the tasks may not pose a problem to some students (Yeo, 2015). The following shows an example of a problem-solving task.
Problem-Solving Task: Handshakes
At a workshop, each of the 64 participants shakes hand once with each of the other participants. Find the total number of handshakes.

2.2 What is a mathematical investigation?

However, some educators believe that a problem must have a clearly defined goal (e.g. Henderson & Pingry, 1953; Orton & Frobisher, 1996). For example, the above Problem-Solving Task has a clearly defined goal: find the total number of handshakes. But the following investigative task does not have a clearly defined goal because the word ‘investigate’ is an ill-defined goal, i.e. what do you investigate?

Investigative Task 1 (Open): Happy Numbers
Choose any positive integer. Square each digit of the number and add to obtain a new number. Repeat this process for the new number. Investigate.

Although researchers agree that there are overlaps between problem solving and investigation, Yeo and Yeap (2009a) observed that many of them still end up separating them into two distinct activities: investigation must entail open investigate tasks with an open goal (e.g. Orton & Frobisher, 1996) and an open answer (e.g. Pirie, 1987); while problem solving is limited to closed problem-solving tasks with a closed goal and a closed answer (e.g. Evans, 1987). On the other hand, some educators believe that investigation encompasses both problem posing and problem solving (e.g. Cai & Cifarelli, 2005).

The issue lies with a widespread adoption of the term ‘investigation’ as the task itself when investigation is actually a process (Ernest, 1991). Therefore, Yeo and Yeap (2010) believed that there is a need to distinguish between investigation as a task, a process and an activity. As a task, most researchers prefer it to be open, just like the above Investigative Task 1. But as a process, students can solve problems by investigation (Yeo & Yeap, 2009b) when they specialise and generalise, formulating and proving conjectures.
For example, for the above Problem-Solving Task (Handshakes), students can investigate by examining smaller numbers of participants, such as how many handshakes there are for 2 participants, 3 participants, and so forth (this is specialising), in order to try to generalise. Along the way, they may find a pattern and formulate a conjecture. Then they may try to prove the conjecture. Specialising, generalising, conjecturing (i.e. formulating conjectures) and justifying (i.e. proving conjectures) are the four main mathematical thinking processes advocated by Mason, Burton and Stacey (1985, 2010).

However, another approach to solve the handshake problem is by using deductive reasoning without any investigation. For example, students can reason that the first participant will shake hand with 63 other participants, the second participant will shake hand with 62 other participants, and so forth. Hence, there are generally two approaches to problem solving: the inductive approach (i.e. the investigative process) and the deductive approach.

In other words, as a process, investigation is a subset of problem solving. But this seems to contradict what was said earlier in this section that some educators believe that investigation encompasses both problem posing and problem solving, i.e. problem solving is a subset of investigation. To resolve this issue, Yeo and Yeap (2010) suggested that if we view investigation as an activity involving an open investigative task, then there will not be any ambiguity. For example, to attempt the above Investigative Task 1 (Happy Numbers), students can pose specific problems to solve (i.e. problem posing), and then attempt to solve it (i.e. problem solving). In other words, investigation, as an activity, involves both problem posing and problem solving.

To summarise, there is a need to distinguish between the task, the process and the activity. A problem-solving task has a closed goal and answer while an investigative task has an open goal and answer (Evans, 1987; Orton & Frobisher, 1996). A problem-solving activity using a problem-solving task usually employs the problem solving process, which consists of two general approaches: the inductive approach (i.e. the investigative process) and the deductive approach; while an investigative activity using an investigative task usually utilises processes such as problem posing and problem solving (Yeo & Yeap, 2010). But why is
problem solving not enough? Why is there a need for mathematical investigation?

2.3 Why should students engage in mathematical investigation?

Many educators (e.g. Civil, 2002) believe that mathematics classrooms should reflect what academic mathematicians do in their workplace. Since it is difficult to bring the mathematical content of academic mathematicians’ practices down to the level of primary or secondary school students, another way is to involve students in a variety of rich mathematical activities that parallel the processes that mathematicians engage in. But what are some of these processes? The bottom line is that mathematicians do not just solve problems but they actively seek out problems to solve.

Lampert (1990) believed that these rich mathematical activities should empower students to think mathematically, such as problem posing (Brown & Walter, 2005), and the four main mathematical thinking processes of specialising, conjecturing, justifying and generalising (Mason et al., 1985, 2010). Jaworski (1994) also viewed an investigative approach to mathematics teaching as involving the four main mathematical thinking processes when she observed that “making and justifying conjectures was common to all three [investigative] classrooms, as was seeking generality through exploration of special cases and simplified situations” (p. 171).

Hawera (2006) claimed that the use of open investigative tasks can help students focus on “the process of problem solving and the open-endedness of a problem or investigation” (p. 286). Students will have to formulate their own problems to investigate or to solve. This will help them to be more aware of the problem situation and to take charge of the issue at hand.

In real life, whether in our workplace or in our personal life, many genuine problems are open and ill-structured in nature. Although these problems are mostly not mathematical, it may be possible to use far transfer to apply mathematical processes to solve the problems. For example, we may realise that something is wrong in real life but we may not know what the real problem or the root cause is. No one is going to clearly define the problem or specify the boundaries of the problem for us
(Simon, 1973). Thus, just like problem posing in mathematical investigation, there is a need for us to formulate the problem first, or to find the root cause, before we can resolve the issue.

After finding the problem, how do we solve it? We may have to examine specific evidence, just like specialising in mathematical investigation where we analyse special cases or specific examples, in order to think of a way to solve it. In real life, the solution may or may not be a generalisation. Along the way, we may have to form some theories or conjectures as to what have gone wrong or how to solve the problem. We may even have to test some of these conjectures to see if they are correct or if they work, just like testing conjectures in mathematical investigation.

Thus we see that the thinking processes behind mathematical investigation are not very different from the 21st century competencies needed to solve workplace or real life problems. The issue is of course whether students are able to transfer these mathematical processes to other domains in real life. But if teachers do not teach their students these processes and make the link to real life, the latter may not realise that it is possible to apply mathematical thinking processes to solve other types of problems which are not mathematical in nature.

Therefore, it is important to expose students to problem posing through the use of open investigative tasks, in addition to problem solving. The next section will discuss how this could be done through the use of an exemplar to illustrate the various mathematical processes.

3 Open and Guided Investigative Tasks

This section will begin with the use of an open investigative task to explain how teachers can develop in their students different types of mathematical processes, followed by a discussion on how more scaffolding could be provided to students with no experience in this kind of investigation by converting the open investigative task into a guided investigative task. The section will end with a discourse on how mathematical investigation could be conducted in the classroom.
3.1 Open investigative tasks

Investigative Task 1 (restated below) will be used to exemplify how teachers can use an open investigative task to empower their students to think mathematically, in particular, problem posing and the four main mathematical thinking processes advocated by Mason et al. (1985, 2010).

**Investigative Task 1 (Open): Happy Numbers**
Choose any positive integer. Square each digit of the number and add to obtain a new number. Repeat this process for the new number. Investigate.

The first question is whether students know how or what to investigate when given this kind of open investigative task. I have discovered that many students in Singapore, and even trainee teachers, are not familiar with this kind of open mathematical investigation, so they could not even start (Yeo, 2008a, 2008b, 2014).

Thus there is a need for teachers to guide their students how and what to investigate. They could use the Model for Cognitive Processes of Mathematical Investigation, developed by me, as a guide (Yeo, 2013). The model is shown in Figure 1 and it will be called the Investigation Model in this chapter. It is not advisable to show such a complex framework to students at the start. Instead, teachers can teach their students each stage of the investigation activity in the model, and each process in each stage, by using Investigative Task 1, which will be illustrated next, before showing the students the model at the end as a consolidation.
Figure 1. Model for cognitive processes of mathematical investigation
The first stage of an investigation activity is to understand the task, just like Pólya’s (1957) model for problem solving. I have found out that some students misinterpreted that ‘the new number’ specified in the third sentence of the task statement of Investigative Task 1 refers to a completely new random number when it actually refers to ‘a new number’ in the second sentence (Yeo, 2013). Therefore, it is very important for teachers to get their students to understand the information given in the task statement correctly.

The next question is to understand what the task means by the word ‘investigate’. Teachers can explain to their students that the goal of such tasks is find any underlying patterns or structures, so the students can just pose the general problem of searching for any patterns under the second stage of problem posing. (In the Investigation Model, students can also pose specific problems to investigate, which I will look at later in this section.) After students have understood that the goal of such tasks is to search for patterns, I have discovered that many of these students did not actually pose this general problem explicitly during thinking aloud in my research study, but they just proceeded directly to the next stage of specialising in order to search for patterns (Yeo, 2013).

The idea of specialising is to examine special cases or specific examples (Mason et al., 1985, 2010). For Investigative Task 1, teachers can guide their students to follow the instructions given in the task by choosing a few positive integers to investigate.

The fourth stage is conjecturing. Teachers can guide their students to search for any patterns. More importantly, there is a need for students to know that any observed patterns may not be true. In theory, an observed pattern is a conjecture to be proven or refuted. But in practice, I have discovered that some students, who knew that observed patterns are only conjectures, did not straightaway treat observed patterns as conjectures (Yeo, 2013). Rather, when they first observed a pattern, they would go back to the stage of specialising to try more examples, to see if this pattern happens by chance. If it is, with more examples, the students may realise that this pattern can be rejected by empirical data, i.e. the observed pattern is false. If not, the students may then consider the observed pattern as a conjecture to be proven or refuted.
For example, for Investigative Task 1, a student may start with the number 7 and generate the following sequence based on the given instructions: 7, 49, 97, 130, 10, 1 (i.e. it terminates at 1). Then the student may start with another number, say, 28, and generate the following sequence: 28, 68, 100, 1 (i.e. it terminates at 1). Novice investigators may straightaway treat this observed pattern as a conjecture to be proven or refuted, but I have discovered that the more experienced investigators will usually try a couple more examples to see if this is really a pattern first, before they formulate it as a conjecture (Yeo, 2013). In fact, other starting numbers, such as 15 and 33, will not result in terminating sequences, but sequences that end in a loop of eight numbers (highlighted in bold in Figure 2). The numbers in Figure 2 are called sad numbers, while the numbers that terminate at 1 are called happy numbers. Of course, we do not expect most students to know these names, and it does not matter that they don’t know. But for ease of discussion, these names will be used.

\[
\begin{align*}
15 & \rightarrow 26 & 65 & \leftarrow 18 & \leftarrow 33 \\
\downarrow & & \downarrow & & \\
62 & \rightarrow 40 & 61 & \leftarrow 56 \\
\downarrow & & \downarrow & & \\
11 & \rightarrow 2 & \rightarrow 4 & \rightarrow 16 & \rightarrow 37 & \rightarrow 58 \\
\uparrow & & & & \downarrow & \\
20 & \leftarrow 42 & \leftarrow 145 & \leftarrow 89 & & \\
\uparrow & & & & & \\
& & & & 154 & 
\end{align*}
\]

*Figure 2. Sequences of sad numbers*

At this stage, the students may formulate the conjecture that all positive integers either terminate at 1 or enter into that loop of eight numbers. The next stage is justifying. In theory, students should try to justify their conjecture by using a formal proof or a non-proof argument.
But in practice, I have discovered that some students would decide to try more examples to see if they could refute their conjecture by empirical data (Yeo, 2013). In a way, this is different from rejecting their observed pattern by empirical data because in this stage of justifying, the students have already formulated their conjecture, while in the previous stage of conjecturing, they were still trying to firm up their observed pattern.

This stage of justifying a conjecture may not be easy for many students, especially if the conjecture requires a complicated formal proof. For example, the above conjecture that all positive integers either terminate at 1 or enter into that loop of eight numbers is not easy to prove at the level of secondary school students. What the teacher can do is to get the students to investigate other things first. But the teacher should come back to teach the students how to justify other conjectures that are within their abilities to prove.

Now, let us consider the next stage of generalising. Actually, the purpose of specialising is to generalise, and so generalising, as a process, cuts across the stages of conjecturing and justifying. However, in the Investigation Model, after justifying a conjecture, there are only two possible outcomes at this stage: is it a generalisation or not? If students can prove that all positive integers either terminate at 1 or enter into that loop of eight numbers, then it is a generalisation that we can divide positive integers into two distinct groups: happy numbers and sad numbers. If a conjecture is false, there is no generalisation.

The seventh stage is checking. Teachers can stress the importance of checking their solution, whether it is correct or whether it makes sense.

At this juncture, students can go back to the task and ask what else there is to investigate. They may continue with the general problem of searching for more patterns or they may pose specific problems to solve. For Investigative Task 1, students may continue with the general problem because they may have observed other patterns during the first round of specialising. For example, from Figure 2, students may have observed that it does not matter if the digits of a number are arranged in a different order (e.g. both 26 and 62 will lead to the same subsequent term 40), or if zeroes are inserted into the number (e.g. 4 and 40 will lead to the same subsequent term 16). This will lead to the conjecture that the rearrangement of the digits of a number, or the insertion of zeroes anywhere between the digits
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of a number, will not change the state of the number (the state refers to whether the number is happy or sad).

What else is there to investigate? At this juncture, students can go back to the second stage of problem posing and start posing more specific problems, such as the following.

Q1. Are there infinitely many happy numbers and infinitely many sad numbers?
Q2. Are there more happy numbers or more sad numbers?
Q3. Is the sum of two happy numbers always happy? Is the sum of two sad numbers always sad?
Q4. Is the product of two happy numbers always happy? Is the product of two sad numbers always sad?
Q5. Which year(s) in this decade is/are happy?
Q6. When two zapping sequences first merge at the same number (e.g. 100), do the preceding terms before this number always have the same unique combination (e.g. 68, 86, 608, 680, 806, 860, 6008, etc.)?

To answer Q1, students can use other heuristics (third stage in the Investigation Model), such as deductive reasoning, to argue that there are infinitely many happy numbers and infinitely many sad numbers because we can insert any number of zeroes between the digits of a happy number or a sad number. Nevertheless, for most questions, the students will still have to resort to specialising, e.g. students can find counter examples for Q3, Q4 and Q6. For Q5, students can specialise to solve the problem without generalising (sixth stage in the Investigation Model).

For Q2, students can also specialise to discover that there are more sad numbers than happy numbers from 1 to 100. This may suggest that there are more happy numbers than sad numbers, but could this be generalised to all positive integers? I do not know the answer to this question. Teachers must be prepared that they do not know the answer to every question when it comes to authentic mathematical investigation. They could perhaps explain to students that there are currently many unsolved maths problems in the world, and that some maths problems may be easy to pose but difficult to solve.
Finally, we come to the last stage in the Investigation Model called extension. The idea of extension is to change some of the givens in the task statement (Brown & Walter, 2005; Kilpatrick, 1987). For example, instead of squaring each digit of a positive integer, what happens if we cube each digit? Or instead of adding the squared digits of a positive integer, what happens if we find the product of the squared digits? This will change the original task and its underlying patterns to a completely new task with very different structures. Alternately, the students may end the investigation for the day, and perhaps come back to it on another day.

3.2 Guided investigative tasks

If teachers feel that Investigative Task 1 is too open for their students who have no experience in this kind of investigation, another approach is to give the students a guided investigative task. How can we rephrase Investigative Task 1 so as to provide more scaffolding? The following shows an example.

Investigative Task 2 (Guided): Happy Numbers

(a) Square each digit of the number 68 and add to obtain a new number: $68 \rightarrow 6^2 + 8^2 = 100$. Repeat this process for the new number 100 until you have a good reason to stop. Why do you stop?
(b) Start with another number 62 and do the same thing until you have a good reason to stop. Why do you stop?
(c) 68 is called a happy number while 62 is called a sad number. Without doing any calculation, explain whether 86 and 608 are happy or sad numbers.
(d) What else can you investigate?

The first three parts provide some guidance for the students while part (d) opens up the task for the students to investigate whatever they want. Part (d) is very important because teachers should not close the task. If part (d) is removed, this task will become what is called a guided-discovery task. In guided discovery, students are guided to explore special
cases in order to discover a formula, a procedure or a mathematical fact which the teacher has in mind (Bruner, 1961; Collins, 1988). In other words, guided discovery is closed in terms of its goal and answer, unlike investigation, which is open (Ernest, 1991).

For this kind of guided investigative tasks, students can still use the Investigation Model (see Section 3.1 Figure 1) to guide their investigation, but they will have to skip the second stage of problem posing at the beginning for parts (a) to (c) in Investigative Task 2 (Guided) because the problems are given in these three parts. However, they will have to go back to the problem posing stage for part (d).

### 3.3 How to implement mathematical investigation in the classroom

For students who have no experience with open investigation, or for novice investigators, it may be advisable for the teacher to start with guided investigative tasks before progressing to open investigative tasks. The teacher can use the Investigation Model as a guide to teach the students the different stages and processes of mathematical investigation. Because there are so many different stages and processes, it may not be advisable to teach all of them at the same time. Rather, the teacher can focus on one or two main processes for each task.

For example, for Investigative Task 2 (Guided), the first three parts are just warm up questions for students to do some investigation because the question is stated in each part and the students do not have to choose their own examples to specialise. But for part (d) on what else to investigate, the teacher can use the four numbers in the first three parts, namely 68, 62, 86 and 608, to guide the students to see if they can use these four examples to generalise. In other words, the teacher is teaching the students that one of the problems that they can investigate is to see if they can generalise from some specific examples. The teacher can then introduce to students the formal terminologies for three of the processes: problem posing, specialising and generalising. But the main focus is still on problem posing in this case.

Later on, the teacher can focus on specialising and conjecturing, or on what other types of problems the students can investigate, for the same Investigative Task 2. Depending on the abilities of the students, the teacher
will have to decide when to progress to open investigative tasks, after only one or perhaps a few guided investigative tasks. For a more detailed description of a programme that introduces open mathematical investigation to secondary school students, the reader can refer to Yeo (2013).

However, some educators (e.g. Jaworski, 1994; Mason, 1978) are worried that teachers may teach mathematical investigation in an algorithmic manner by stereotyping certain mathematical processes as a set of procedures to be learnt by students. For example, Lerman (1989) observed a lesson by an experienced teacher who taught mathematical investigation by telling his students what to do to arrive at an answer when they were stuck. Thus a task that is intended to be open can be closed by the teacher in its implementation.

Therefore, it is very important not to tell students the solution when they are stuck. Otherwise, the students would not be doing the investigation. Instead, when students are stuck, the teacher can ask guiding questions to stimulate thinking and further investigation. For example, if students are stuck in part (c) of Investigative Task 2 (Guided), the teacher can guide the students by asking, “What do you notice about the digits in the numbers 68 and 86?” Sometimes, the students just need more time to investigate and think things through, so the teacher should not try to rush it.

4 More Illustrating Examples of Investigative Tasks

This section will incorporate a few more examples of investigative tasks that teachers can use in class. If the tasks are open, the reader can think of how to rephrase them to convert them into guided investigative tasks. Since the processes have been exemplified in Section 3 for Investigative Task 1 (Happy Numbers), they will not be deliberated in this section for the following tasks. The section will instead focus on what to investigate, and whenever possible, link the content to the school syllabus, such as common multiples and prime numbers. In addition, an important misconception for Investigative Task 3, even among teachers, will be dealt with.
4.1 The year 2016

In Section 3.1, the answer to Q5 is 2019: it is the only year in this decade that is happy. There are other interesting years to investigate. If possible, we should try to use years that are more recent. The following shows a guided investigative task involving 2016, which was the year at the time of writing.

Investigative Task 3 (Guided): The Year 2016
The year 2016 is divisible by 1, 2, 3, 4, 6, 7, 8 and 9. When is the next year that has the same properties? What else can you investigate?

A common mistake made by students, and even some teachers (based on anecdotal evidence), is that the answer to the first question in the above task is $2016 \times 2 = 4032$. To appreciate why this is wrong, we will look at the correct answer first.

The smallest number that has the same properties is the LCM of 1, 2, 3, 4, 6, 7, 8 and 9. Students are familiar with finding the LCM of two or three numbers, but do they know how to find the LCM of eight numbers? Other than by using brute force to find the LCM of the eight numbers at the same time, there is a shortcut in this case. Notice that the LCM of 1, 2, 4 and 8 is 8; and the LCM of 1, 3 and 9 is 9. Since 8 and 9 are relatively prime, the LCM of 1, 2, 3, 4, 8 and 9 is just the LCM of 8 and 9, which is equal to $8 \times 9 = 72$. Since 6 is a factor of 72 (because $6 = 2 \times 3$, and both 2 and 3, which are relatively prime, appear in the LCM of 1, 2, 3, 4, 8 and 9), then the LCM of 1, 2, 3, 4, 6, 8 and 9 is the same as the LCM of 1, 2, 3, 4, 8 and 9, which is equal to the LCM of 8 and 9. Lastly, since 7 is relatively prime to 1, 2, 3, 4, 6, 8 and 9, then the LCM of 1, 2, 3, 4, 6, 7, 8 and 9 is just the LCM of 7, 8 and 9, which is equal to $7 \times 8 \times 9 = 504$. Therefore, the correct answer is $2016 + 504 = 2520$. 
But do students know why we must add the LCM to 2016 to obtain the next year with the same properties? The first five common multiples of 1, 2, 3, 4, 6, 7, 8 and 9 are:

- \(504 = \text{LCM}\)
- \(1008 = \text{LCM} \times 2\)
- \(1512 = \text{LCM} \times 3\)
- \(2016 = \text{LCM} \times 4\)
- \(2520 = \text{LCM} \times 5\).

It is clear from the above list of common multiples that the next year with the same properties as 2016 is the next common multiple after 2016, i.e. 2520, and the difference between 2016 (\(\text{LCM} \times 4\)) and 2520 (\(\text{LCM} \times 5\)) is just the LCM. This is why we add the LCM 504 to 2016 to obtain the next common multiple.

From the above list of common multiples, we also observe that the next common multiple after 504 is actually \(504 + 504\), or \(504 \times 2 = 1008\). Now we come back to the common mistake mentioned earlier in this section: the reason why the next common multiple after 2016 is not \(2016 \times 2 = 4032\) is because 2016 is not the LCM, unlike 504.

Students have learnt how to list common multiples and how to find LCM. But examination questions usually involve LCM and not common multiples. Therefore, Investigative Task 3 is a good investigation to empower mathematics learners to develop a deeper understanding of common multiples, and the relationship between LCM and common multiples, namely, the difference between consecutive common multiples is the LCM, and only the second common multiple is double the first common multiple. Since common multiples and LCM are already in the syllabus, we are not teaching entirely new concepts that are outside the syllabus, but we are strengthening students’ relational understanding of existing constructs.

However, there is a bigger issue here, even for maths teachers. During a workshop conducted by me during the Mathematics Teachers’ Conference in Singapore in 2016, a few teachers contended that the next year which has the same properties as 2016 is not 2520 because 2520 is also divisible by 5, but 2016 is not divisible by 5. The main question is,
“What do you mean by having the ‘same properties’?” This is a real issue for all teachers because their students may also raise the same objection and then they will have to give a convincing reply.

Before we answer this question, let me explain that the next common multiple after 2520, which is $2520 + 504 = 3024$, is also not the answer because 3024 is also divisible by 432, but 2016 is not divisible by 432. Similarly, the next common multiple after 3024, which is $3024 + 504 = 3528$, is also not the answer because 3528 is also divisible by 49, but 2016 is not divisible by 49. If we continue like this, then the only number that has the ‘same properties’ as 2016 is 2016 itself.

There are two ways to address this issue. First, the issue is the confusion as to what constitutes a property of a number. We all agree that ‘divisible by 5’ is a property, so the teachers who raised the concern said that 2016 does not have this property, so the next year should not have this property also. On the other hand, the teachers did not realise that ‘not divisible by 5’ is also a property. The task did not state that 2016 is not divisible by 5, so when the task asks for the next year with the ‘same properties’ as 2016, the ‘same properties’ do not include this property called ‘not divisible by 5’. In other words, we must not insist that the next year should have this property called ‘not divisible by 5’. Thus, it does not matter whether or not the next year is divisible by 5.

The second way to look at this is that it is perfectly fine for the next year to have other properties that 2016 does not have. For example, if we define a parallelogram to have two pairs of parallel sides, then what is another quadrilateral that has the ‘same property’ as a parallelogram? An answer is a rectangle. But one may argue that this is wrong because other than having two pairs of parallel sides, a rectangle also has right angles which a parallelogram does not (necessarily) have. However, we know that a rectangle is also a parallelogram, except that a rectangle has other extra properties such as right angles. In other words, when the question asks for a quadrilateral with the ‘same property’ as a parallelogram, it does not mean that the quadrilateral cannot have other properties which the parallelogram has or does not have.

Another more real life example would be, “I am a teacher. Are you a teacher?” You may say, “Yes, I am a teacher.” But I say, “No, you are not a teacher because you are single (compare: 2520 is divisible by 5), but I
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am not (compare: 2016 is not divisible by 5)” or “You are a female, but I am not.” Of course, no one would reason like this by adding other attributes to a teacher that are not mentioned in the original assertion that “I am a teacher.” This real life example also attempts to show how logic used in maths can be related to the logic used in real life. If we are not careful with the logical reasoning used in maths, we may end up making illogical reasoning in real life.

Finally, what else can we investigate for this task? The following are just some suggestions:

(a) When was the previous year that had the same properties?
(b) What is the smallest positive integer with the same properties?
(c) When is the next year that is divisible by 1, 2, 3, 4, 5, 6, 7, 8 and 9?
(d) What other properties do 2016 have? Investigate.

From the list of common multiples above, the answers for parts (a) and (b) are 1512 and 504 respectively. Part (c) is a natural question since the only divisor missing from that long list of divisors is 5. From the list of common multiples again, the answer to part (c) is 2520 because 2520 = LCM(1, 2, 3, 4, 6, 7, 8, 9) × 5. Since 5 is relatively prime to all the other divisors, LCM(1, 2, 3, 4, 5, 6, 7, 8, 9) = 2520. In fact, 2520 is also divisible by 10, so the smallest positive integer that is divisible by 1 to 10 is 2520. If we want 11 to be a divisor also, then the number will have to be much bigger because 11 is relatively prime to all the other divisors, i.e. the smallest positive integer that is divisible by 1 to 11 is 2520 × 11 = 27720.

For part (d), students may have to search the Internet. One property we will discuss in the next section is that 2016 is a triangular number.
4.2 Polygonal numbers and binomial coefficients

The following shows another investigative task.

Investigative Task 4 (Open): Triangular Numbers

The figure below shows a sequence of triangular numbers, so called because the dots for each term can be arranged in the shape of a triangle. Investigate.

\[
\begin{align*}
T_1 &= 1 \\
T_2 &= 3 \\
T_3 &= 6 \\
T_4 &= 10
\end{align*}
\]

For this kind of tasks, a standard problem to investigate is whether there is a general formula for the \(n^{th}\) term of the sequence. By examining each case (specialising), students should infer that \(T_n = 1 + 2 + 3 + \ldots + n\).

Although secondary school students have not learnt the formula to find the \(n^{th}\) term of an arithmetic progression, they should have learnt in primary school how to find, say, \(1 + 2 + 3 + \ldots + 100\), by doing the following:

\[
\begin{align*}
1 + 100 &= 101 \\
2 + 99 &= 101 \\
3 + 98 &= 101 \\
&\quad \vdots \\
50 + 51 &= 101
\end{align*}
\]

\[
\therefore \ T_{100} = 1 + 2 + 3 + \ldots + 100 = 101 \times 50 = 5050
\]
If \( n \) is odd, the above procedure needs to be modified a bit because the middle term will be the odd one out when pairing the numbers. But for general \( n \) terms, it would be easier to do the following:

\[
\begin{align*}
1 + n &= n + 1 \\
2 + (n - 1) &= n + 1 \\
3 + (n - 2) &= n + 1 \\
&\vdots \\
n + 1 &= n + 1 \\
\therefore 2T_n &= n(n + 1) \\
T_n &= \frac{n(n + 1)}{2}
\end{align*}
\]

Teachers may also wish to link these numbers to the problem of finding the total number of handshakes if \( n \) people are to shake hand once with each other (see Problem-Solving Task in Section 2.1). Since the first person will shake hand with \( n - 1 \) people, the second person with the remaining \( n - 2 \) people, etc., the total number of handshakes will be \( (n - 1) + (n - 2) + (n - 3) + \ldots + 3 + 2 + 1 \), i.e. \( 1 + 2 + 3 + \ldots + (n - 1) = T_{n - 1} \), the \((n - 1)\)th triangular number.

What else can we investigate? In line with my fascination with interesting years in this decade, we can investigate which years in this decade are triangular numbers. As mentioned at the end of Section 4.1, 2016 is a triangular number. But for students who do not know this, solving \( \frac{n(n + 1)}{2} = 2001 \) to \( \frac{n(n + 1)}{2} = 2010 \) can be tedious and time consuming. Is there any trick that they can use? We can empower students by guiding them to use approximation: \( \frac{n(n + 1)}{2} \approx \frac{n^2}{2} \), so \( \frac{n^2}{2} \approx 2001 \) will give \( n \approx 63.3 \); when \( n = 63 \), \( T_{63} = 2016 \). We can also find the previous year and the next year that are triangular numbers: \( T_{62} = 1953 \) and \( T_{64} = 2080 \), which I will not live to see it.

Since secondary school students who take Additional Mathematics would have come across Pascal’s Triangle and Binomial coefficients,
teachers may guide them to observe that $T_{n-1} = \frac{n(n-1)}{2} = \binom{n}{2}$. This means that all the Binomial coefficients $\binom{n}{2}$ and $\binom{n}{n-2}$ are triangular numbers since $\binom{n}{2} = \binom{n}{n-2}$.

But what other interesting Binomial coefficients can we investigate? Again, in line with my obsession with interesting years, other than $\binom{64}{2} = 2016$, are there recent years that can be expressed as $\binom{n}{r}$, where $r \neq 1$ or $n - 1$? (If $r = 1$ or $n - 1$, it is nothing remarkable since $\binom{n}{1} = \binom{n}{n-1} = n$ will give you any positive integer that you want.) The most recent one is $\binom{14}{5} = \binom{14}{9} = 2002$. The next year that this will occur is in $\binom{24}{3} = \binom{24}{21}$ = 2024, while the previous year that this happened was in $\binom{63}{2} = \binom{63}{61}$ = 1953 (the latter is a triangular number).

Other than triangular numbers, students can also investigate square numbers (which are just perfect squares), pentagonal numbers, hexagonal numbers, etc., to find a general formula for the $n^{th}$ term of each type of numbers. Then the students can generalise one more level up by finding a general formula for the $n^{th}$ term of any $m$-gonal numbers, which is $T_n = \frac{n[(m-2)n + (4-m)]}{2}$. 
4.3 Other special maths years and days

In Section 4.2, we have seen that 2016 is a triangular number, while in Section 3.1, we have seen that 2019 is the only year in this decade that is happy. What other special maths years or days are there for students to investigate?

Investigative Task 5 (Guided): Prime Years
(a) Twin primes are prime numbers that differ by 2. Find out the names of prime numbers that differ by 4, and those that differ by 6.
(b) Investigate which years in this decade are prime numbers. Are they twin primes or is there a name for this kind of primes?

For part (a), students can search the Internet to learn that prime numbers that differ by 4 are called cousin primes and those that differ by 6 are called sexy primes (because the Latin word for six is sex). For part (b), students can use trial division to discover that 2011 and 2017 are the only prime years in this decade, and since they differ by 6, they are sexy prime years. In other words, the year of publication for this book, 2017, is a sexy prime year.

Below is another example of an investigative task on special maths years and days.

Investigative Task 6 (Guided): Perfect Square Year and Square Root Day
(a) A perfect square or square number is an integer that is the square of an integer. For example, 0 (= 0²), 1 (= 1²), 4 (= 2²) and 9 (= 3²) are perfect squares. Which years in this century are perfect squares? How often do perfect square years occur? Investigate.
(b) A square root day occurs when both the day and the month are the square root of the last two digits of the year. For example, the previous square root day was on 4/4/16.
When is the next square root day? How often do square root days occur? Investigate.

For part (a), 2025 (= 45²) is the only perfect square year in this century since the previous one was 1936 (= 44²) and the next one is 2116 (= 46²). When perfect squares are small, e.g. 0, 1, 4 and 9, they are closer together, but as they get bigger, they are further apart, until the present day when it occurs only once in a century. For part (b), the next square root day is 5/5/25. There are 9 square root days in each decade, from 1/1/01 to 9/9/81. Some students may wrongly include dates like 0/0/00 and 10/10/100.

Below is another example of an investigative task on special maths days.

Investigative Task 7 (Guided): Pi Day, Pi Approximation Day and Tau Day
(a) The circumference of a circle with a diameter of 1 unit is π units. Explain why Pi Day is observed on March 14 and Pi Approximation Day is observed on July 22. Investigate.
(b) The circumference of a unit circle (i.e. a circle with a radius of 1 unit) is 2π units. As a result, some people believe that 2π is more fundamental than π. So they define τ (pronounced as tau) as equal to 2π (notice that the symbol τ looks like the symbol π). Explain why Tau Day is observed on June 28. Investigate.

For part (a), since the first 3 digits of π is 3.14, and \( \frac{22}{7} \) is an approximation of π, Pi Day and Pi Approximation Day are observed on March 14 and July 22 respectively. Students can investigate when the first official celebration of Pi Day was and how people usually celebrate. They can also find out other interesting properties of π. For part (b), since the first 3 digits of τ is 6.28, Tau Day is observed on June 28. Students can investigate in more details why some people consider τ to be more
fundamental than \( \pi \). They can also find out why June 28 is also called Perfect Day, which we will discuss in the next section.

4.4 Other special numbers

René Descartes, a French mathematician, once said this, “Perfect numbers, like perfect men, are very rare.” The first two perfect numbers are 6 and 28, which explains why June 28 is also called Perfect Day. In fact, all positive integers are either perfect, abundant or deficient. The following is an example of an investigative task on these numbers.

Investigative Task 8 (Guided): Perfect Numbers, Abundant Numbers and Deficient Numbers

(a) Proper factors of a number are factors that are less than the number itself, e.g. the proper factors of 6 are 1, 2, and 3. The aliquot sum of a number is the sum of all its proper factors, e.g. the aliquot sum of 6 is \( 1 + 2 + 3 = 6 \). If the aliquot sum of a number is equal to itself, the number is called a perfect number, e.g. 6 is the smallest perfect number. Find the next larger perfect number. Are perfect numbers rare? Is there a formula to generate perfect numbers? Investigate.

(b) If the aliquot sum of a number is greater than itself, it is called an abundant number. If the aliquot sum of a number is less than itself, it is called a deficient number. Investigate.

For part (a), the next four perfect numbers after the number 6 are 28, 496, 8128 and 33 550 336. Since the fifth one already have 8 digits, it does suggest that perfect numbers are rare. A formula to generate perfect numbers is \( 2^{p-1}(2^p - 1) \), where \( 2^p - 1 \) is a Mersenne prime. (Numbers in the form \( 2^p - 1 \), where \( p \) is prime, are called Mersenne numbers. If \( 2^p - 1 \), where \( p \) is prime, is also prime, then it is called a Mersenne prime.) The Euclid-Euler theorem states that every even perfect number can be represented in the form \( 2^{p-1}(2^p - 1) \), where \( 2^p - 1 \) is a Mersenne prime. It is not known whether odd perfect numbers exist or not. Another
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investigation that students can do will be on Mersenne numbers and Mersenne primes, e.g. which Mersenne numbers are primes?

For part (b), students can first find out some examples of abundant and deficient numbers. For example, 12 is an abundant number because its aliquot sum, \(1 + 2 + 3 + 4 + 6 = 16\), is greater than the number itself; while 15 is a deficient number because its aliquot sum, \(1 + 3 + 5 = 9\), is less than the number itself. With more examples, students may discover that for a number to be abundant, it must have many factors (i.e. an abundant number of factors) so that its aliquot sum can be greater than itself. Similarly, a deficient number must have fewer factors. Other things to investigate include whether all positive integers are either abundant, perfect or deficient, whether all prime numbers are deficient, and which years in this decade are abundant or deficient. (For info, the year at the time of writing, 2016, is an abundant number since its aliquot sum 4536 is greater than itself.)

It is beyond the scope of this section to discuss other types of special numbers. But the reader may be interested to find out more about the following types of numbers: polite numbers (and the degree of politeness), lucky numbers, safe primes, Sophie Germain primes, self-descriptive numbers and Erdös–Nicolas numbers (for info, the year 2016 is the second smallest Erdös–Nicolas number).

### 4.5 Non-arithmetical investigation

So far, all the investigative tasks we have looked at are on numbers. Although it is beyond the scope of the chapter to look at mathematical investigation in other topics, I will leave the reader with a geometric investigation so that he or she may have an idea of what can be done for other mathematics topics. This investigation can be used to develop the Midpoint Theorem for triangles.

Investigative Task 9 (Open): Midpoints of Quadrilateral

Draw any quadrilateral. Join the midpoints of all the sides of the quadrilateral to form another quadrilateral. Investigate.
5 Conclusion

Andreas Schleicher, head of the OECD’s education assessment programme, was quoted in the Financial Times Magazine as saying, “Mathematics in Singapore is not about knowing everything. It’s about thinking like a mathematician.” (Vasagar, 2016) Other than thinking how to solve problems, mathematicians also pose problems to investigate and to solve. Therefore, open and guided mathematical investigative tasks can be very useful for teachers to use in the classrooms to empower their students to think mathematically, and to formulate, investigate and solve problems like mathematicians. Moreover, mathematical processes that are developed from doing mathematical investigation may also be translated to solve non-mathematical problems in the workplace and in real life, although far transfer is not easy for many people to develop.

References


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Chapter 13

Using Representations to Develop Mathematical Thinking

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Representations used in mathematics instruction can be divided into either external or internal systems of representations. External representations involve the use of concrete models, diagrams, ICT, verbal, real world and symbolic representations to exemplify and illustrate mathematical concepts. Internal representation refers to how a mathematical idea is constructed and represented in the minds of the learner by means of personal notation systems, natural language, visual imagery, and problem solving strategies. In choosing representations for instruction, a teacher has to ask whether a particular model and materials work; how do we know they work; how should they be used; what do students need to learn from their use, and how will this be communicated. Such questions in the use of representations are related to teaching and learning, and will help teachers to think about what students might “see” in a representation, and provide a glimpse of the internal representations students create as they interact with external representations. Carefully selected representations help the teacher show the mathematics to the pupils the teacher is trying to teach. Good teaching goes beyond just using physical representation to the thinking skills that has to be abstracted from the representations used by the teacher.
1 Introduction

One of the major aims of mathematics teaching and learning for the 21st century is for the development of mathematical thinking (Devlin, 2014) and there is a widespread agreement that school mathematics should be taught as a thinking activity. The common misconception among many math purist and even math educationist (including teachers) is that “doing mathematics” is the same as developing “mathematical thinking”. If learning is seen as a matter of adding, or stockpiling new concepts to existing ones then there is little room for thinking on the part of the learner. But on other hand if learning is viewed as a constructive process which leads to changes in the learners existing schemas, then learning can be seen as involving the learners thinking processes. What pedagogical skills and tools must teachers possess to facilitate this aim? There is a concern from education authorities all over the world to address this aim as 16 percent of the 21st century has already passed.

In order to develop students’ mathematics thinking repertoire for today’s learning, Devlin (2014) suggests that, as educators, less attention should be on procedural learning that focused on computational skills and procedures and more on guiding students “learn how to learn” and develop “a good conceptual understanding of mathematics, its power, its scope, when and how it can be applied, and its limitations” (p. 21). A key to achieving this focus is for teachers to pay more attention to student’s voice by listening to them diligently and value their contribution. In other words, to facilitate this, teachers need to be provided with knowledge and examples of thinking tasks activities that showcase how they may encourage the development of mathematical thinking among learners during mathematics lessons. We are of the view that these thinking activities can be developed through deliberate use of representations during mathematics lessons.

2 The Use of Mathematical Representations as a Tool (to Develop Mathematical Thinking)

What are representations? “A representation is a configuration that can represent something else in some manner” (Goldin, 2002, p. 208).
Historically people have developed and used representations in order to interpret, remember or communicate their experiences. From early cave dwellers to the Egyptians hieroglyphics and the Babylonian numeration systems, we see the use of representations in the expression and communication of ideas.

2.1 Types of mathematical representations

Numerous research studies and position papers have been written on this subject over the years. Lesh, Landau, and Hamilton (1983) found five kinds of representations that are useful for mathematical understanding: (a) real life experiences, (b) manipulative models, (c) pictures or diagrams, (d) spoken words, and (e) written symbols. Other have specifically looked at representation used to understand mathematics (Goldin, 2003; Goldin & Shteingold, 2001; Lesh, Landau, & Hamilton, 1983; Lesh, Post, & Behr, 1987). In many cases access to abstract mathematical ideas and concepts can only be obtained through the representations of those ideas and concepts through diagrams, charts and symbols (Kilpatrick, Swafford, & Findell, 2001). According to Bruner people represent ideas in three distinct forms, (a) through actions - enactive), (b) through pictures or diagrams - iconic and, (c) through words and language - symbolic. (Bruner, 1966).

Goldin and Shteingold (2001) suggest a useful way of classifying representations. They suggest two systems of representation - external systems of representation and internal systems of representation. External representations in mathematical context refers to quantitative cognitive tasks that are usually quantified or measurable such as computing the four fundamental operations (add, subtract, multiply and divide) with paper and pencil, drawing a graph, writing algebraic expression, communicating verbal mathematical language and etc. where these cognitive task has been built over a period of time. On the other hand, internal representation refers to qualitative cognitive tasks that goes in the mind of an individual with the goal of giving meaning to it. Examples of internal representation includes personal notation systems, natural language, visual imagery, and problem-solving strategies (Salkind, 2007). These two types of representation by Goldin and Shteingold are of great relevance in the
context of this chapter and will be discuss them in more depth in sections 2.2 and 2.3.

2.2 External representations

External representations can be further divided into purposeful representations and contextual representations. Purposeful representational tasks involve teachers’ use of concrete models, diagrams, pictures, ICT, verbal and symbolic representations not only to exemplify the mathematics but also to illustrate the mathematical concepts. Such tasks are associated with good mathematics teaching (Watson & Mason, 1998). The mathematical purpose is clear and the models and representations are linked directly and explicitly to the mathematical concepts the teacher wishes to convey. The representations are intended to “show” mathematics to the learners (Fennel, 2006) through the use of manipulative materials (teaching aids), diagrams, graphical displays and symbolic expressions. Yackel (1984) introduced the term "external code" to refer to the following: diagrams, symbolic notation, equations, and charts or lists. Contextual representations situate mathematics within a contextualized practical problem to show how mathematics is used to make sense of the world and to motivate students to solve the problem.

2.3 Internal representations

Internal representations refer to how a mathematical idea or concept is constructed and represented in the minds of the learner. It also includes internalizing or taking in mathematical ideas through personal notation systems, natural language, visual imagery, problem solving strategies and understanding them. We refer to these as mental representations. Our brain organizes knowledge into knowledge structures called schemas (Piaget, 1970; Skemp, 1979). These range from simple to complex. Simple schemas become more complex as the learner learns more about a concept mediated through two internal cognitive processes - assimilation and accommodation. Another important mediating process in the development of internal representation is scaffolding. According to Vygotsky (1978), when a teacher or a more knowledgeable person assists a child through
instruction, questioning, prompts, and cues the teacher move the child to a state of understanding in which internal representations are constructed.

The use of both these representations in mathematics context has been of special interest to mathematics educators. Good teaching goes beyond just using physical representation to the thinking skills that has to be abstracted from the representations and incorporating them into appropriate schemas internally thereby developing efficient internal systems of representations that connect to established external systems of representation (Goldin, 2002). Figure 1 depicts how the use of these representations lead to mathematical understanding. External representations used by the teacher must be connected to internal representations that a pupil constructs for the development of mathematical thinking.

Figure 1. Internal and external representation and mathematical understanding
The use of representation as a learning device can definitely be capitalized by teachers to enhance student’s meaningful learning of mathematics. Representation as a tool enhances student’s capabilities in solving more complex problems thus improving their mathematics achievement and motivation. Therefore, it is important to examine the nature of mathematical representations and their uses in mathematics classroom/instruction.

3 Representation in the Mathematics Classroom

As we have seen earlier, learning involves the interpretation of ideas and concepts through the perspective of the learner’s existing knowledge. The student is therefore not seen as passively receiving knowledge from the environment; although instruction clearly affects what children learn, it does not determine it, because the child is an active participant in the construction of his own knowledge. It is at this juncture that the use of representations becomes critical in the learning of concepts and ideas. Appropriate representations are chosen by the teacher, in classroom instructions in order to make meanings clear for the learners so that they may be able to accurately construct knowledge. Numerous research studies have found representations to be powerful aids to student learning (Cuoco & Curcio, 2001; Flevares & Perry, 2001; Goldin & Shteingold, 2001; Kilpatrick, Swafford, & Findell, 2001; Veloo & Lopez-Real, 1994, Veloo, 1996). Representations are also used in the learning of new mathematical concepts and in clearing up students’ mathematical confusions (Flevares & Perry, 2001; Parmjit, 2004).

3.1 Representation and mathematics instruction

Ball, Thames and Phelps (2008) argues that effective mathematics instruction requires more than pedagogical content knowledge (PCK). She argues that mathematical knowledge for teaching include the following:

- Knowing mathematics content;
- Knowing which concepts are easy or difficult to learn and why;
• Knowing ways of representing concepts so that others can understand them;
• Knowing how to connect ideas to deepen them; and
• Recognizing what students might be thinking or understanding.

Mathematical knowledge for teaching is a deep understanding of mathematics (Ma, 1999) that allows teachers to assess their students’ work, recognize both the sources of student errors and their students’ understanding of the mathematics being taught. They also can appreciate and nurture the creative suggestions of talented students. Additionally, these teachers see the links between different mathematical topics and make their students aware of them. Teachers with deep understanding are also more able to excite students about mathematics (CBMS, 2001).

The ability to use representations in mathematics instruction to make meanings clear is an important part of teachers’ Mathematical Knowledge for Teaching. Representations have traditionally been used in the mathematics classrooms to support mathematical reasoning, enable mathematical communication and convey mathematical thought (Kilpatrick, Swafford, & Findell, 2001). Unfortunately, the use of representations in instruction is often limited to external systems of representation which students use to solve mathematical problems, and these forms of external representations are often seen as an end in itself (Goldin, 2002). In order to understand the mathematical concept that is being conveyed by the external representation, the learner has to process it internally connecting the mathematical ideas and concepts to appropriate internal knowledge structures or schemas (Piaget, 1970; Skemp, 1979). A learner’s failure to make this connection often results in misconceptions that manifest itself in the form of various computational errors (Olivier, 1989).

3.2 Children’s use of representations

Research evidence obtained from studies carried out in Malaysia and Brunei Darussalam (Veloo & Lopez-Real, 1994; Veloo, 1996, Parmjit 2004) suggests that many learners do not readily use external representations such as diagrams or concrete models in solving problems.
When asked why they did not draw a picture or model the problem before solving it, their answer was, “that is not mathematics”. This is also due to the fact that the teacher do not use these forms of representations as a natural problem solving strategy. Children often model procedures used by teachers during instruction (Bandura, 1986) and teachers do not readily use external representations during classroom problem solving. Teacher demonstrations in classes are mainly the internal thinking processes (internal representations) of teachers in solving problems. In the three studies quoted above (Veloo & Lopez-Real, 1994; Veloo, 1996; Parmjit, 2004), we found that pupils’ success in solving word problems improved when they used pictorial representations as a functional tools for problem solving or as a way to represent their thinking to work out the solution. An example of the representation is as follows in Figure 2:

![Figure 2](image)

*Figure 2. Concrete (blocks) to Pictorial (example of number bond) to Abstract*

They need to move from a concrete representation of the problem to a symbolic form. Taking pupils from concrete representations to symbolic representations through teacher led discussion and questioning leads to the development of mathematical thinking. Given the importance of representations in the teaching and learning of mathematics, we address three fundamental questions, Questions 1 to 3 in the following subsections, related to the use of representations in schools.

**Question 1**
How can the teacher promote the use of representations as a natural problem solving and thinking strategy, in solving problems, among elementary school children?
In the study by Veloo and Lopez-Real (1994) and Veloo (1996) it was found that, out of the 693 responses that were analysed, only 35 (or 5%) of the children used a diagram spontaneously to solve the given word problem. This indicated that although diagram drawing as a modelling strategy has been advocated in the literature, in reality it is seldom used by children in solving problems. The children needed assurance from the teacher that the use of representations (diagrams) is an accepted problem solving strategy in doing mathematics. The assurance can be in the form of verbal reinforcement during problem solving process; reinforcements can also come from teacher modelling the use of diagrams and models in solving problems in the classroom. Regular teacher modelling can serve to reinforce this approach to problem solving. In using these models, teacher need to verbalize the thinking strategy that is used in conceptualizing the relationships between the variables in the problem. In doing so the teacher makes his or her thinking visible to the learners.

In the three studies Veloo & Lopez-Real (1994), Veloo (1996) and Parmjit (2004), it was found that both suggestions that a diagram could be drawn and explicit instructions on how problem could be represented in the form of diagrams, resulted in increased success in solving word problems. The following example illustrates helpful diagrams that Year 6 pupils used to represent the problem before solving it.

*Problem 1: A teacher brought a bunch of bananas to be shared equally among 12 boys. Eight of the bananas were bad and were thrown away. After this each boy received 4 bananas. How many bananas were in the bunch at the start? (Veloo & Lopez-Real, 1994)*

*Figure 3. Students’ pictorial representation of the problem*
Problem 2: Seven girls share 3 pizzas and 3 boys share one pizza. Who gets more pizza, the girls or the boys? (Parmjit, 2004)

![Image of student's reasoning on pizza item]

**Figure 4.** Student’s reasoning on the pizza item

**Question 2**
Are there stages in the use of representations in solving school mathematics problems?

Figures 5, 6 and 7 which follow were drawn by Year 7 pupils to represent problems 3, 4 and 5 respectively after they were given explicit training by the researchers on how to represent a problem in the form of a diagram and linking the diagram to an algebraic representation. The “model method” popularly used in Singapore schools was taught to pupils prior to the test. The results show that students represented the world problems in three different ways, which we refer to as category (a), category (b) and category (c).
Problem 3: In a class three-fifths of the pupils were girls and the rest were boys. If the number of boys were doubled and 6 more girls joined the class there will be equal number of boys and girls. How many pupils were in the class at the start?

Figure 5. Students’ reasoning on the classroom problem
Problem 4: In a class of 40 pupils, 25% were girls. When some new girls joined the class, the percentage of girls increased to 40%. How many new girls joined the class?

Figure 6. Students’ reasoning on the percentage problem
Problem 5: Three tired and hungry men went to sleep with a bag of apples. One man awoke, ate one-thirds of the apples and then went back to sleep. Later a second man awoke and ate one-third of the remaining apples and went back to sleep. Finally the third man awoke and ate one-third of the remaining apples. When he finished there were 8 apples left in the bag. How many apples were in the bag at the start?

Figure 7. Students’ reasoning on the apple problem
These results lend support to the approaches recommended by theorists and practitioners that problem solving should move from the concrete, to the semi-concrete before linking them to abstract representations in the form of algebraic structures. Rushing learners too early into symbolization was found to be not helpful. Even in drawing diagrams the teacher should bear in mind that the diagram drawn is not abstract. The example below illustrates this point.

In our studies, we found that a sizable proportion of pupils were unable to solve problem 1 using the authors’ representation of the problem. The authors' diagrams were all at the symbolic end of the continuum (the authors’ diagram for problem 1 is shown in Figure 8). It could well be the case that some pupils were not ready for such representations and yet might have been able to interpret a more physical or concrete version as evidenced by the diagrams drawn by some of the pupils themselves. From the results of our studies we believe that pupils needed to progress through stages of development in modelling the problem using representations for problem-solving. The stages of development should begin from a concrete representation of the problem and eventually moving to a symbolic representation of the problem. This approach moving from concrete representations to symbolic algebraic representations through teacher led discussion and questioning leads to the development of mathematical thinking.
Question 3
How can teachers promote mathematical thinking through the use of representations?

In many countries, one of the important aims of school mathematics is to develop mathematical thinking and thinking skills. The important question facing the mathematics teacher is how can this be brought about in the regular mathematics class? Pupils face greater difficulty in mathematics when compared to other subjects in the school curriculum. One of the reasons is due to the early introduction of symbolic and abstract mathematical concepts for which many children are least prepared. Pupils definitely need to be helped through active hands-on activities through the use of representation. These learning activities empower pupils to take ownership of their own learning that is meaningful for them. For example, as shown in Figure 2, the hands-on experiences of concrete (fruits) to pictorial (number bond) to abstract allow pupils to understand how numerical symbols and equations operate at a concrete level, making it more meaningful for them. In a nutshell, the use of representations enhances pupils' reasoning and problem-solving abilities; help them make connections among ideas; and aids them in learning new concepts and procedures. The use of representation makes it possible for teachers and pupils to represent abstract mathematical ideas and concepts in different ways. This makes it possible for pupils to see and think about an idea from different angles, which in our view inevitably leads to the development of mathematical thinking.

3.3 Exemplification /Function of external representations

Figures 3 and 4 are examples of diagrams or external representations of the problem made by the learners after suggestion by the researchers when the learners have failed at the Comprehension or Transformation stages of the Newman’s (1977) classification. It is clear that the external representations helped in some way to overcome those difficulties. Two elements of understanding are involved in these instances. The first arises from the Newman instruction “tell me what the question is asking you to
do.” This question effectively asks whether the pupil understands the structure of the problem and does not probe any understanding of the second element which are the relationships between the variables involved in the problem. Thus, it was clear that some pupils understood the problem in terms of the structure but not the relationships involved. The act of making an external representation of the problem focuses the learners’ attention on what the numbers represent. In this sense the function of an external representation may be to act as an alternative form of “expressing the problem in one’s own words.” Having focused on the meaning of the numbers and their relationships within the problem, it then appears that external representation can act as a key aid in the Transformation stage of solving the problem. The diagram forces the learner to think more deeply about the relationships among what is known and unknown in the problem. Pupils who faced difficulty at this stage could be given prompts by the teacher through questions.

3.4 Exemplification /Function of internal representation

Solutions to the Pizza problem, shown in Figure 9, exemplify pupil’s internal representations.

<table>
<thead>
<tr>
<th><strong>Level 1</strong></th>
<th><strong>Level 2</strong></th>
<th><strong>Level 3</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Iteration of ratio units (through addition)</td>
<td>Invariant of the ratio unit (coordination of two number sequences)</td>
<td>Scalar functional operator</td>
</tr>
<tr>
<td>2 students : 3 pizzas</td>
<td>2 students : 3 pizzas</td>
<td>students pizzas</td>
</tr>
<tr>
<td>2 students : 3 pizzas</td>
<td>4 students : 6 pizzas</td>
<td>2 : 3</td>
</tr>
<tr>
<td>2 students : 3 pizzas</td>
<td>6 students : 9 pizzas</td>
<td>x 5</td>
</tr>
<tr>
<td>2 students : 3 pizzas</td>
<td>8 students : 12 pizzas</td>
<td>10 : 15</td>
</tr>
<tr>
<td>2 students : 3 pizzas</td>
<td>10 students : 15 pizzas</td>
<td>x 5</td>
</tr>
<tr>
<td>Symbolic representation:</td>
<td>The ratio unit of 2 : 3, and the composite unit of 10 : 15 was iterated in terms of 2 : 3</td>
<td>It is a scalar and consists of transposing the operator that links 2 to 10 in students (in this case a scalar of 5) to the other measure space pizzas, and then applying it to 3 to get 15.</td>
</tr>
<tr>
<td>10 : 15 = (2 : 3) + (2 : 3) + (2 : 3) + (2 : 3) +</td>
<td>10 : 15 = (2 : 3) + (2 : 3) + (2 : 3) + (2 : 3) +</td>
<td>10 : 15</td>
</tr>
</tbody>
</table>

Figure 9: Stages of internal representations
The solutions show the transitional thinking methods that bridge addition, and multiplication to proportional reasoning. We refer to these methods as "iterative multiplication" representations. Students use this "iterative multiplication" process of thinking in each domain both to conceptualize the underlying problem situation and to carry out the numerical solution processes.

The ways in which internal representation are expressed in pupils’ conceptual structure (scheme) is fundamental to how teachers can use these schemes in their teaching. They will be able to bridge the gap from pupil’s naïve solution strategies to more formal and sophisticated ones. As illustrated above, a pupil should know how to represent the iteration of ratio units through addition, followed by invariant iteration of the ratio unit. As pupils become more knowledgeable, they will develop a repertoire of formal representation, such as scalar functional operator and to use them meaningfully. Written representations of mathematical ideas are important tools in learning mathematics as this will enhance and organize the learners’ thinking. Even if an idea seems to lead nowhere, it is still important for the teacher to encourage the learner to represent and to reflect on them. Reflecting on the relative strengths and weaknesses of various representations leads to thinking about the problem from various angles and this again is mathematical thinking in action.

4. Representations and Mathematical Thinking

In our investigation involving school pupils we have found that the type of external representations used by pupils increased in sophistication as they matured. Figures 5, 6 and 7 are examples of different types of representations used by pupils. The representations in category (a) were more abstract in nature whereas those in category (c) were more concrete in nature. This has implications for teachers intending to use diagrams (external representations) as a problem-solving strategy. Young children need to first experience more concrete representations of a problem. This can then be followed up by semi-concrete and later into more abstract forms of representations as their thinking capabilities advance. The results of these studies strongly suggest that representations made by pupils in
which they are able to incorporate relational aspects of the problem provide a definite aid to a successful solution of a problem. Internal representations of learners in solving problems also increase in sophistication from additive to multiplicative and then to proportional reasoning. These processes we believe are evidences of mathematical thinking in action.

In choosing representations for instruction, the teacher has to ask himself/herself several questions. Will a particular model and material work? How do we know they work? How should it be used? What do students need to learn from its use, and how will this be communicated? These questions related to teaching and learning will help teachers to think about what students might “see” in a representation, such as a concrete model or a drawing that they create in solving a problem. They provide a glimpse of the internal representations students create as they interact with any representation. Carefully selected representation helps the teacher show the mathematics to the pupils the teacher is trying to teach by making mathematical ideas and concepts clear to their students (Fennel, 2006). Good teaching goes beyond just using physical representation to the thinking skills that has to be abstracted from the representations used by the teacher. This is done by explanations, posing questions and soliciting answers from pupils as well as giving opportunities for pupils to ask questions. These activities make thinking visible; the teacher’s thinking becomes visible to the pupil and the pupil’s thinking becomes visible to the teacher (Fennel, 2006). When planning instruction, practice, or reinforcement activities, teachers should consider how they and their students can use representation in today’s mathematics lesson.

5 Conclusion

In this chapter, we have examined the use of representation in the context of primary school mathematics. Numerous research studies have reported on the benefits of the use of representations in teaching mathematics. Researchers generally agree that representations are useful in the teaching and learning of mathematics. If judiciously used it makes meanings clear to pupils. One of the purposes of using external representations in
Representations and Mathematical Thinking

mathematics instruction is that its use must lead to the development of appropriate internal representations. Correct internal representations lead to mathematical understanding and problems solving. Incorrect internal representation lead to misconceptions.

In realizing this objective, it is important for teachers to draw out the mathematical ideas embedded in the representations during classroom discussion and activities. This enable pupils to engage meaningfully and to actively take part in the interactive processes of teaching and learning (Barwell, 2005). A salient approach is the one that encourages pupils to ponder deeply about the mechanics and process of mathematical thinking upon completing a task, which would usually fabricate cognitive tools that the pupils could use fluidly in various random circumstances (Wiggins, 1998). However, it should be noted that teacher’s deep knowledge about mathematical concepts and the use of representations are important factors in pupils learning as it impacts their ability to use representations effectively.

How can teachers empower mathematics learners during mathematics lessons? Teachers should listen to the voices of pupils and be willing to unlearn and relearn to learn pupils’ way of thinking; "Who dares to teach must never cease to learn." (Quotation - John Cotton Dana in 1912). Empowering students helps us all do just that. Secondly, they should shift their approach from the traditional computation and routine based instruction to a conceptual one. The former method involves teaching of rules and procedures rather than thinking skills. Mathematical thinking shall be developed by getting pupils to think about mathematics and representing concepts and ideas in ways other than procedures. The development of strong mathematical thinking skills will give pupils more confidence in approaching problems and more importantly, it will empower them to take ownership of their own learning. Therefore, we strongly recommend the use of mathematical representations as a tool to develop mathematical thinking.
References


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Chapter 14

Empowering Teachers to Use Open-Ended Real-World Tasks in Primary Mathematics Classrooms

NG Kit Ee Dawn

This chapter reports the findings from an exploratory study which investigated (a) what teachers perceived as features of real-world tasks, (b) whether teachers perceive real-world tasks as useful platforms to engage students meaningfully with the mathematics they learn, (c) teachers’ mathematical reactions to open-ended real-world problems as solvers, and finally (d) the tensions teachers face with regards to authenticity of context, task complexity, and expectations of students’ work from open-ended real-world problems. Two open-ended real-world problems designed for primary school students were used as platforms to elicit teachers’ responses. Findings reveal that teachers have concerns about assumption making and real-world interpretations associated with the problem contexts. In particular, they surfaced challenges related to the design and facilitation of such tasks to engage children in meaningful mathematical learning. Implications from the findings are drawn for teaching, learning, and teacher education so as to suggest the way forward for empowering teachers in using open-ended real-world tasks in primary mathematics classrooms.

1 Introduction

The Singapore mathematics curriculum framework (Ministry of Education, 2012) advocates the use of a ‘variety of problems, including
open-ended and real-world problems” (p. 17). This is to create opportunities for students to select appropriate school-based mathematical knowledge and skills to help them make reasonable informed decisions within real-world constraints during problem solving. Furthermore, the incorporation of applications and mathematical modelling in the Singapore mathematics curriculum framework since 2007 have prepared students for the activation of mathematical literacy in the 21st century as espoused by the Organisation for Economic Co-Operation and Development [OECD]

Mathematical literacy is an individual’s capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens. (OECD, 2015, p. 5)

The stage is further set for real-world problem solving when “Problems in Real-World Contexts (PRWC)” assessment items are formally introduced in the 2016 GCE “O” Levels examinations (Singapore Examinations and Assessment Board [SEAB], 2016).

Given the increasingly crucial role of real-world problems in teaching and learning in Singapore mathematics classrooms, it is not surprising that local research in this area has picked up momentum in recent years (e.g., Chan, 2014; Lee & Ng, 2015; Ng, Widjaja, Chan, & Seto, 2015). Nonetheless, both international and Singapore research on real-world problem solving have mainly concentrated on (a) the analysis of students’ mathematical responses in real-world tasks (e.g., Busse, 2005; Chan, 2010; Ng, 2011; Stillman, Brown, & Galbraith, 2010), (b) the impact of real-world tasks on students’ perceptions of the relevance of school-based mathematics to real-life experiences (e.g., De Corte, Verschaffel, & Greer, 2000; Ng & Stillman, 2015; Sáenz, 2009; Smith & Morgan, 2016), (c) the nature of task design and pedagogical issues with implementation of real-world tasks (e.g., Tan & Ang, 2013; Kawakami, Saeki, & Matsuzaki,
Empowering Teachers: Open-Ended Real-World Tasks

2015), and (d) teacher development in use of real-world tasks for teaching and learning (Kaiser & Grünewald, 2015; Tan & Ang, 2015; Widjaja, 2013). Existing studies have surfaced both the potentials and challenges of implementing real-world tasks in mathematics classrooms all over the world. Findings on the effects of such tasks on mathematical empowerment (Ernest, 2002) and engagement have been inconclusive because of the myriad task designs and pedagogical approaches used by teachers.

Research analysing the success of educational reforms (e.g., Hargreaves, 1998) highlighted teacher beliefs and attitudes as critical factors. Indeed, teachers’ perceptions of using open-ended real-world tasks would govern the extent to which they harness the potentials of such tasks for sustained implementation focusing on assisting students in making meaningful connections between school-based mathematics and the real-world. Nonetheless, investigations into teachers’ perceptions about the use of real-world tasks in Singapore mathematics classrooms have been few and far between to date. This chapter reports an exploratory qualitative study which builds on a previous one (see Ng, 2010) in the continual journey to empower teachers in using open-ended real-world problems in Singapore primary mathematics classrooms.

2 The Role of Contexts in Open-Ended Real-World Problems

Real-world problems are generally problems situated in authentic contexts where solvers are required to integrate appropriate contextual, conceptual, and procedural knowledge in their solution process so as to propose reasonable mathematical responses to the problem at hand which are within real-world constraints. Sáenz (2009) articulated that contextual knowledge is “related to everyday-life problems in the real world” and that it “enters the school situation through the presentation of the problem in a context with its own story” (p. 126). In other words, when students activate their contextual knowledge during real-world problem solving, they immerse in the context presented in the problem by taking on perspectives within the given context during their interpretations of the problem and subsequent choices of appropriate mathematical approaches. Ideally,
problem solvers should be “integrating” where they perceive the real-world problem in its real context and apply their real-life experiences about the context to help “mathematise and validate the solution” (Busse, 2005, p. 355). However, like Busse and others, Ng (2011) discovered that students very often made limited or weak connections between the mathematics and their real-life experiences when evaluating the reasonableness of their solutions in an authentic real-world problem. More often than not, real-world constraints related to the context of the problem are overlooked or excluded in students’ solutions. Verschaffel, Greer, and De Corte (2000) called this the “suspension of meaning” (p. 3). On the other side of the coin, little is known about teachers’ expectations or perceptions of “integrating” behaviours in their students when they design and select real-world tasks for teaching and learning. Much less is known about how teachers can be empowered to do so.

Clarke and Helme (1996) proposed two views about the use of contexts in real-world problems: (a) context as referred to but bounded by task requirements and (b) the real-life context from which the task is crafted out of. The former resonates with what we know as “applications” because the rationale of teacher incorporation of context in the design of such tasks is for students to apply the chosen mathematical knowledge and skills taught earlier (see Stillman & Brown, 2011). The latter aligns with “mathematical modelling” activities where modellers mathematise the real-world situation through use of appropriate mathematical lenses in cyclical stages of formulating, solving, interpreting, and reflecting of model development (see CPDD, 2012) striving towards multiple revisions of the model so as to obtain a sophisticated, robust solution of the problem. Yet, the question of how authentic should the context of school-based problems be is difficult to answer. Clarke and Roche (2009) cautioned that although real-world problems are meant to serve twin purposes of “showing how mathematics is used to make sense of the world and motivating students to solve the task” (p. 25), the chosen context has to be meaningful to the students. In addition, teachers should engage students to “nail the mathematics” (p. 29) in such tasks so that learning within the context could be more mathematically fruitful. Nevertheless, one may still argue that the meaningfulness of the context can be subjective from student to student, given different life experiences. The OECD (2015)
proposes the use of four meaningful contexts in real-world problems that young people encounter in their lives: personal, societal, occupational, and scientific (p. 5). Furthermore, the authenticity of a context not only relates to the meaning ascribed to it by the students, it also depends on how the problem is presented to the students. Often, certain features or real-world constraints of the context are truncated when the problem is presented to students bearing pedagogical and curriculum considerations.

How the real-world task is presented to students leads to the question of the degree of open-endedness in task design that teachers choose. Indeed, the articulation of the purpose, the given information, and the conditions of the problem tend to frame the degree of open-endedness in a problem. These, together with the nature of the context, allow for a range of student interpretations during assumption making as part of the solution process. Singapore curriculum documents use the term “open-ended” but Yeo (2015) refers this as the “openness” of a task. He classified the openness of a mathematical task based on five task variables. Extending from Orton and Frobisher (1997), Yeo discusses two spectrums when describing task goals: open vs closed goals and well-defined vs ill-defined goals. Open goals simply refer to situations where students have to set their own goals for the task. In contrast, closed goals are already defined in the task. Well-defined goals mean clarity in conveying task goals to students whilst ill-defined goals require students to make further interpretations of the possible goals within task requirements and even set their own parameters in the solution process. On the other hand, Frobisher (1996) proposed perceiving the openness of a task through its solution process; whether there are multiple ways of solving or just one way. Yeo calls this open-method vs closed-method of solving. Lastly, Wood (1986) outlined task complexity as a function of the “relationships between task inputs” and surfaced it to be “an important determinant of human performance through the demand it places on the knowledge, skills, and resources of individual task performers” (p. 66). Yeo suggests that teachers can scaffold students’ attempts at the task so as to relieve some of the task complexity for students.

Nonetheless, limited research with experienced mathematics teachers revealed teacher tensions when balancing between just-in-time guidance for students to help them get unstuck, and allowing opportunities for
students to frame and subsequently review their approaches during open-endedness real-world tasks (Lee, 2013; Ng et al., 2015). Chan (2013) also surfaced low levels of confidence expressed by experienced mathematics teachers in Singapore schools when “dealing with openness in task interpretations and solutions” (p. 411). In an earlier study, Ng (2010) highlighted teacher insecurities about accepting a non-linear problem solving process and their challenges with interpreting and evaluating alternative solution pathways in open-ended real-world problems. Subsequently, Ng (2013) highlighted the need to empower teachers so that they are ready to incorporate real-world problems in their mathematics classrooms, particularly open-ended ones. She proposed that teacher educators work towards cultivating a mind-set change in Singapore mathematics teachers so as to develop a positive climate that fosters discussions of different mathematical representations and solutions to the same problem arising from multiple interpretations of the context. Given the recent curricula focus across the world and major national assessment demands in relation to real-world problems of varying open-endedness in Singapore, among other countries, teacher education research in this area is still at infancy. Research on teacher education in the use of open-ended real-world tasks have mainly concentrated on pre-service teachers; specifically their attitudes towards such tasks, their task interpretations involving the use of real-world or contextual knowledge, and their mathematical solutions (e.g., Kaiser & Grünewald, 2015; Tan & Ang, 2015; Widjaja, 2013). To date, little is known from the ground, directly from experienced primary mathematics teachers who are given the key role of providing firm foundations in students’ mathematical learning and perceptions of mathematics: (a) what they perceived as features of real-world tasks, (b) whether they perceive real-world tasks as useful platforms to engage students meaningfully with the mathematics they learn, (c) their mathematical reactions to open-ended real-world problems as solvers, and finally (d) the tensions they face with regards to authenticity of context, task complexity, and expectations of students’ work from open-ended real-world problems.
3 Method

This study sets out to investigate (a) to (d) above during a 3-hour interaction session between the author and 14 primary school teachers in Singapore. Each participant had at least one year of teaching experience in schools. It is hoped that the findings from this exploratory qualitative study serves to inform teacher educators and teachers the focuses of subsequent teacher professional development courses on the use of open-ended real-world tasks in teaching and learning.

3.1 Data collection and analysis procedure

At the beginning of the session, the teachers recorded their thoughts in writing on what they perceived as features of real-world tasks and whether they perceived real-world tasks as useful platforms to engage students meaningfully with the mathematics they learn. Subsequently, the teachers attended a one-hour introductory lecture on the different types of real-world tasks and the four kinds of contexts (i.e., personal, societal, occupational, scientific) they could use when designing real-world tasks. During the lecture, the teachers were given opportunities to ask questions and discuss the various degrees of open-endedness in the various tasks provided. In the next hour, the teachers worked in pairs or groups of threes to solve one out of three open-ended real-world problems designed for Primary Three, Four, and Five students respectively. Every group recorded their solution on paper along with assumptions, conditions, and variables considered. Throughout the group discussions, the author went from group to group, taking field notes of the queries and concerns of the teachers as they worked on the problems. In the third and final hour of the session, each group made an oral presentation of their solution and reflected on their reactions to open-endedness of the problem both as solvers and as teachers who may facilitate these problems in their own primary mathematics classrooms. They were asked to comment on the problem design as well as to articulate any tensions they may face with regards to the authenticity of context, task complexity, and their expectations of students’ work arising from the problem. These comments and reflections were noted by the author. The mathematical approaches used for each
problem were analyzed along with the teachers’ written responses and the researcher’s notes. Two of the problems and findings related to them will be elaborated below.

3.2 The Granny’s Rug problem

Adapted from Downton, Knight, Clarke, and Lewis (2006) who used this problem with grades 5 to 8 students, the Granny’s Rug (Figure 1) required solvers to interpret the context, make assumptions about the shape and size of the hallway, and set conditions before they propose different ways they could cut up the carpet to fit in the hallway. One of the most common assumptions is that the hallway is in a rectangular shape. An important condition is that there should be no wastage of material.

Granny’s Rug

My granny bought a square rug for her hallway.
Each side of the square rug measured 1 m.
When granny reached home, she found that the rug would not fit in the hallway. She decided to cut the rug and join the pieces together to make the shape that would fit the hallway.
All of the rug must be used.
What are the different ways the rug looks like now?

Figure 1. The Granny’s Rug task

This problem was chosen because it has an “open method” (Yeo, 2015) as there are multiple ways of interpreting how the carpet can be cut. The mathematical content knowledge required by the problem can include fractions, decimals, and area. Solvers have to bear in mind that regardless of how they were to cut up the rug, the total area of the rug still remains as 1 m². For instance, the carpet can be cut into two equal parts and joined to form a 2 m by 0.5 m rectangular shape or it can be cut into four equal strips and joined to form a 4 m by 0.25 m elongated rectangular shape. Advanced solvers can discuss how they devise systematic ways of forming composite shapes with a total area of 1 m². Mathematical reasoning is involved when solvers discuss which composite shapes are more appropriate for the
hallway (e.g., rectangular shapes) and which are not feasible to cut (e.g., shapes with dimensions that are in recurring decimals such as 0.33 m).

### 3.3 The Water Walker Ball problem

The Water Walker Ball problem (Figure 2) was inspired by a chance encounter by the author on the following YouTube link: [https://www.youtube.com/watch?v=o3eN4OWreD4](https://www.youtube.com/watch?v=o3eN4OWreD4) where children and adults use walker balls in pools for physiotherapy purposes. The Water Walker Ball scenario is a very authentic context, although not common one in Singapore. This problem was designed for children in Primary Three and above as it involves concept applications such as duration of time, area, and perimeter. The problem is deliberately open-ended, slightly ill-defined but with a known clear goal. There can be many approaches to solve the problems and many possible answers.

<table>
<thead>
<tr>
<th>Water Walker Ball</th>
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</thead>
<tbody>
<tr>
<td>Watch the video.</td>
</tr>
<tr>
<td>You get just 5 minutes at most inside a water walker ball.</td>
</tr>
<tr>
<td>But you may use up the air inside the ball faster if you run too quickly.</td>
</tr>
<tr>
<td>The longest length inside the ball from one end to another is 2 metres.</td>
</tr>
<tr>
<td>The pool is big enough for only 5 balls at one time to roll around freely.</td>
</tr>
<tr>
<td>The pool is open from 9 a.m. to 6 p.m. every day.</td>
</tr>
<tr>
<td>Estimate how many people can use the water walker ball in one day.</td>
</tr>
</tbody>
</table>

*Figure 2. The Water Walker Ball task*

The video not only sets the stage for understanding the use of the water walker ball in real-life, it also helps children to visualise the space within one ball. To make it more fun for the children, the teacher can conduct
activities on measurement in class and have the children role-play the situation. In this way, they can set realistic conditions of use of the pool and make reasonable assumptions from at least two perspectives; the users of the walker balls and the workers manning the use of the water walker balls. For instance, taking on the perspective of a worker manning the use of the walker balls, a solver can set a condition that there should be 5-minute change-over time for the person inside the ball to be replaced by a new person. Another condition may be that the balls are released together in the pool at fixed intervals so as to minimise accidental injuries due to enthusiastic participants. From the perspective of the ball users, one assumption the solver can make is that because the ball seems very fun to play with, all users would most likely take up all the time allocated to use the ball. Of course, we would also assume that all the users would have no breathing issues and that they can last at least 5 minutes inside the ball. The mathematical solutions may not be complicated. Solvers simply had to apply multiplication of whole numbers, calculate the duration between 9 a.m. and 6 p.m., and subsequently work out the number of 5-minute intervals within this duration. It is children’s mathematical reasoning integrated with real-world considerations when applying the concepts that this problem hopes to provide a platform for.

4 Findings

Firstly, the teachers’ general perceptions of the features of real-world tasks and whether they perceived real-world tasks as useful platforms to engage students meaningfully with the mathematics they learn will be presented. Next, the teachers’ solutions to the Granny’s Rug and the Water Walker Ball problems as well as the tensions they faced with regards to authenticity of context, task complexity, and expectations of students’ work from the two problems are discussed in this section.

4.1 Teachers’ general perceptions of real-world tasks

In summary, the 14 teachers perceived real-world tasks to have the following key features:
Empowering Teachers: Open-Ended Real-World Tasks

- authentic
- relevance to daily life
- contain ideas applicable to daily life
- involve real-life problems faced currently or in the future

It is clear that authentic contexts which are relevant to daily life either currently or in the future were the main interpretations of the teachers about real-world tasks. This could suggest that to these teachers, the contexts for the real-world tasks are crucial and contexts should be chosen wisely. Interestingly, there was one teacher who interpreted real-world tasks as “hands-on activities”. It is not known why this narrow understanding of what a real-world task entails could have occurred.

All the teachers also commented that they thought real-world tasks are useful platforms to engage students meaningfully with the mathematics they learn in the different ways below. Figure 3 shows some excerpts from the teachers’ comments. Real-world tasks

- relate to the students so the tasks are engaging;
- help students to connect or apply what they learnt in class to their daily experiences;
- show the purpose in learning mathematics so that students are able to see how the mathematics skills will help them in real-life;
- allow for more meaningful applications of mathematics skills and concepts;
- encourage students to make sense of the mathematics they learn and hence increase engagement level;
- provide student-centred learning to take place; and
- create opportunities for mathematical reasoning and communication.
4.2 Teachers’ solutions to Granny’s Rug problem

Figures 4 and 5 show the solutions to the Granny’s Rug problem from two of the groups. Both groups clearly tried to consider different shapes of the hallway, thus opening up more possibilities for forming composite figures through the cutting of the carpet. Not unexpectedly, both groups also used the expected mathematical concepts and skills this problem was designed for. They approached the problem systematically, organising their mathematical representations such that it was very easy to follow their mathematical reasoning. They all worked within the given condition that no matter how the composite figures are formed, the total area of each figure had to be kept at $1 \text{ m}^2$.

Interestingly, Group 3 seemed to have posed questions about the problem (Figure 4) instead of setting their own parameters for the solution pathways. They were obviously concerned with one key piece of missing information in the problem: the shape of the hallway. They also realised that there is no point in cutting the carpet into narrower and narrower pieces when they progress with their different composite figures as this would not be useful in real-life.
In contrast, Group 4 seemed to have limited the number of composite figures formed to fixed shapes (Figure 5) which they perceived could be reasonable shapes of the hallway. The sizes of the pieces were kept relatively larger than those of Group 3’s because they reasoned that the carpet needs to have enough space to walk on. They verbally commented that not knowing the shape of the hallway was a little challenging at first because based on their life experiences, some residential accommodations may not have hallways and those which do may have hallways in different shapes and sizes. At times, there can be furniture in the hallway and hence the shape of the carpet needs to fit in the remaining space for walking in the hallway.
Teachers’ solutions to Water Walker Ball problem

Groups 5 and 6 managed to get the same answer for the Water Walker Ball problem although they solved it in different ways and made diverse assumptions about the context. The solution from Group 5 (see Figure 6 below) appeared more efficient than that of Group 6’s largely but both groups adopted the same piece of vital information from the problem: the pool can accommodate at most 5 walker balls at one time in order for the balls to roll about easily. Nonetheless, it was conveyed through their
reflections that both groups faced challenges differentiating between the assumptions to be made and the conditions they have to set as part of their solution processes.

The first assumption made by Group 5 (Figure 6) appeared to be the most helpful in getting the solution fast: all 5 balls are used at the same time. The other three assumptions were appropriate based on the context provided and the given information. It is interesting that they came up with three conditions about the logistics of use of the pool although others may question the logic of the last two conditions: it is not economically viable for the company to have 5 balls on standby all the time and it will be impossible for the start time of one shift to be exactly the end time of a previous shift since time is needed for change-over. Based on the first assumption they listed, the group worked out the maximum number of people using the water walker balls in one day within the opening hours of the pool by first calculating that there can be 12 possible “shifts” of users in one hour. Hence, in one hour, there can be 60 users in the 12 shifts. This worked out to be a maximum of 540 users in one day.

The solution from Group 6 (Figure 7) shows the use of systematic listing in a table. Again, proportional reasoning was used albeit in a different manner as Group 5. They calculated that 5 ball users will take 5 minutes in all to finish one shift. There can be 12 shifts taken up by 60 users in one hour. This also led to a maximum of 540 ball users in one day. The assumptions made by Group 6 for this problem are similar to those from Group 5 but they included another piece of information to help them made the assumption about rate of breathing. However, this group did not appear to be able to differentiate between assumptions and conditions in their solution process.
Assumptions:
1. All 5 balls are used at the same time.
2. No one will take less than 5 minutes.
3. Never ending queues.
4. It will not take more than 5 minutes to reach the other end of the pool.

Conditions:
1. The company has at least 10 balls. 5 in use while 5 standby.
2. End time of first session will be the start time for the second session.
3. The pool must be more than 10m long.

Solution:
\[
\begin{align*}
60 \div 5 &= 12 \\
16 \times 12 &= 192 \\
9 \times 5 &= 45 \\
60 \times 9 &= 540
\end{align*}
\]
4.4 Tensions teachers faced with the two problems

At the beginning, the teachers were obviously rather apprehensive about the open-ended nature of the two problems. Particularly the Granny’s Rug problem, the teachers felt that the information provided in the problem was not complete (i.e., the shape and dimensions of the hallway were not provided). Although they were informed that it was a deliberate decision in task design to leave this piece of information out because it would open up more possibilities of different solutions, the teachers would much prefer to provide the information if they were to use the problem in their classrooms.

Both problems had certain levels of authenticity in choice of contexts. For the Granny’s Rug problem, the teachers articulated possible differences in the interpretations of the context between themselves and their students. For instance, in the Granny’s Rug problem, students may not know what a hallway is. This is because the word “hallway” is hardly
used to describe the entrance of Housing Development Board flats which most students may live in. In addition, some teachers questioned about the logic of having a rug of only 1 m² in area as it would seem too small to cover a reasonably-size entrance or hallway of a flat or a house. They also conveyed that they took a long time wondering if the idea of covering the hallway would be similar to carpeting the hallway where every corner and edge of the wall which joins the ground would be covered. Although, the time taken here has delayed their solution process, this time was crucial in making sense of the problem in relation to real-world experiences. In contrast, the groups working on the Water Walker Ball problem had little questions about the context of the problem mainly due to the amount of information already provided for them. They were perceived to be of “equal entry levels” to the context here as compared to the students because of the rare occurrence of Water Walk Ball activities in Singapore. They reflected that the video was very useful in introducing them to the context and it was not difficult to draw meaning from the given context.

The tensions associated with task complexity faced the teachers for the Granny’s Rug problem were mainly related to the open-ended nature of the problem and the teachers’ perceived “incomplete information”. Some teachers found it hard to start working on the problem because of the considerations highlighted above. Others provided mathematical solutions which were not particularly useful in real-life. For example, in Figure 5 where the rug had width of 0.5 m which is rather narrow to walk on, it is unclear whether there was a suspension of meaning for these teachers or they had simply included every mathematical solution they could reasonably think of. For the Water Walker Ball problem, the teachers also found the problem to be complex at first but this was due to the amount of information provided of which they had to decide was crucial to start narrowing possible approaches.

The initial solutions of the teachers for both problems were the mathematical responses they had expected their students to come up with. However, these solutions did not reflect how any assumptions or chosen parameters were used to govern the problem-solving process. It took some prompting by the author to encourage the teachers to articulate their assumptions and parameters for the problem so that there is more
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5 Discussion, Conclusion and Implications

This chapter reports the findings from an exploratory study which sets out to investigate (a) what teachers perceived as features of real-world tasks, (b) whether teachers perceive real-world tasks as useful platforms to engage students meaningfully with the mathematics they learn, (c) teachers’ mathematical reactions to open-ended real-world problems as solvers, and finally (d) the tensions teachers face with regards to authenticity of context, task complexity, and expectations of students’ work from open-ended real-world problems. Two open-ended real-world problems designed for primary school students were used as platforms to elicit teachers’ responses.

The teachers in the sample have identified features of real-world tasks not unlike those defined in literature (e.g., Sáenz, 2009). This could imply that the teachers could be conscious of making careful choices in the contexts in which they situate the real-world tasks. It can be surmised that to the teachers, real-world tasks play a key role in helping students make connections between school mathematics and their daily lives, thus clarifying a central purpose for students’ mathematical learning. They identified the potentials of real-world tasks in fostering meaningful mathematical learning through sense making and mathematical reasoning when working within the authentic contexts. However, at the same time, the teachers also recognised that students need to be able to relate to the real-world tasks in order for these tasks to be useful platforms for student engagement in mathematics. Indeed, Galbraith, Stillman, and Brown (2010) had emphasized “meaningfulness to students” as a central consideration for the design of real-world problems.

The teachers’ solutions to the two tasks has shown that generally, they had activated their real-world knowledge and integrated it rather successfully to come up with mathematically reasonable solutions within the contexts. However, the role of assumptions and conditions in solving open-ended real-world problems seemed to be overlooked by the teachers;
with some teachers unable to distinguish between the two. Assumption making is an important process in solving real-world problems because it is of direct relevance to the choice of mathematical knowledge to be applied and subsequent evaluation of the reasonableness of the solution in view of the context. Both the mathematics syllabus (Ministry of Education, 2012) and the recent set of assessment guidelines (Ministry of Education, 2015) from the Singapore Ministry of Education highlighted the growing prominence in making assumptions explicit in problem solving. Nonetheless, Ng and her colleagues (2015) revealed that teachers have found assumption making to be challenging themselves for open-ended problems and they are faced tensions when asked to scaffold students’ assumption making process.

As reported in other research (e.g., Ng, 2013), the mind-set of teachers in working with real-world problems has to evolve slowly. Real-world problems are often open-ended, ill-defined, and at times this requires the solvers to look for the relevant information themselves. The teachers in the sample were obviously taken aback by the missing information in the Granny’s Rug problem and this had implications on their perceived task complexity of the problem. Furthermore, reflections from Groups 3 and 4 have revealed that although the authenticity of the context allows for integration of real-life experiences with the mathematics applied, these real-life experiences may at times become inhibitions in decision making about the solution pathways when the problem was perceived to be too open-ended or to have lack the necessary information.

The preliminary findings from this exploratory study points out the need for teacher education courses to build on the existing perceptions of teachers on the potentials of real-world tasks in mathematics teaching and learning in the following areas: empowering teachers to (a) unpack the role of assumption making in real-world problems, (b) differentiate between assumptions and conditions, and (c) deal with open-ended problems with perceived “missing information”. In addition, teacher education courses could also consider empowering teachers with knowledge of how to critically analyse solutions which show suspensions of meaning as opposed to the desired “integrating” versions as well as how to move students forward from the former to the latter.
Acknowledgement

The author would like to thank all the anonymous teacher participants of the meeting whose work has provided the content for this chapter.

References


Chapter 15

ACISK Framework – A Tool for Empowering Mathematics Learners to be Self-Directed

WONG Lai Fong  Berinderjeet KAUR

This chapter introduces the ACISK framework, a tool for empowering mathematics learners to be self-directed in their learning. Conceptualisation of the framework is guided by the Singapore school mathematics curriculum and the four phases of problem solving by Polya. The framework may be used by students as a guide for self-questioning, and also to reflect and identify gaps in their knowledge. Most importantly, teachers need to complement student’s use of the framework with other strategies such as giving constructive feedback to empower them in self-regulating and self-managing their learning.

1 Self-directed Learning

In the book, *Self-directed Learning: A Guide for Learners and Teachers* by Knowles (1975), self-directed learning (SDL) is defined as:

- a process in which individuals take the initiative, with or without the help of others, in diagnosing their learning needs, formulating learning goals, identifying human and material resources for learning, choosing and implementing appropriate learning strategies, and evaluating learning outcomes (p. 18).

Self-directed learning views learners as responsible owners and managers of their learning process, integrating self-management (management of the context) and self-monitoring (control of the cognitive)
Empowering Mathematics Learners (Garrison, 1997). Two essential elements of the SDL are: (1) the teacher is the mentor or facilitator rather than the dispenser of knowledge to be deposited into the learner’s memory, and (2) the learner is a thinker and creator of his own knowledge rather than a passive recipient of the knowledge of others (Guglielmino, 2008). In SDL, management and control of learning gradually shift from teachers to learners as learners exercise independence in setting learning goals, decide what is worthwhile learning, and how to approach the learning. (Brookfield, 1986; Hiemstra, 1999).

Why self-directed learning? Guglielmino (2008) gave two basic reasons: (1) our nature as it is our most natural way to learn, and (2) our environment that is complex and is changing at a breath-taking pace (p. 2). Knowles (1975) found that SDL not only allows the learner to learn more deeply and thus commit the learning to long term memory, the learner also develops the disposition for lifelong learning as every experience, in general, affords him an opportunity to learn. Furthermore, to meet the demands of the rapidly changing and increasingly complex work environment, each individual needs to function as a self-directed learner who not only absorbs knowledge, but transfers that knowledge onto the job. Formal education in schools is only a beginning in learning, and it is imperative that students learn to learn and develop life-long learning in order to remain effective.

Given its importance, one way to achieve SDL for learners is for teachers to model learning strategies (Abdullah, 2001), besides making learning meaningful, and to provide learning structures so that learners learn how to learn and develop the ability to use appropriate strategies to learn on their own.

2 The ACISK Framework

In Singapore the framework of school mathematics curriculum, shown in Figure 1, emphasises conceptual understanding, skills proficiency, mathematical processes, attitudes and metacognition. The framework is not only rigorous and robust to reflect the disciplines of mathematics but
also includes the related skills needed to prepare students for the 21st century.

Figure 1. Framework of Singapore School Mathematics Curriculum (Ministry of Education, 2012).

Drawing on the five components (concepts, skills, processes, attitudes and metacognition) of the Mathematics Curriculum Framework, the authors conceptualised the ACISK framework as an empowering tool that students may use to self-direct their learning. It helps them to (1) develop a deep understanding of mathematical concepts, (2) apply the relevant mathematics skills after making sense of the underlying mathematical principles (not merely routinizing procedures), (3) develop metacognition through reflection on their learning process, and (4) boost their confidence in learning mathematics.

The ACISK framework, shown in Figure 2, consists of six questions that mathematics learners have to ask themselves. The questions are as follows:

- What is the problem/question asking for? [A]
- What is/are the concept(s) required? [C]
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- What is/are the given information/condition(s)? [I]
- What is/are the skill(s) required? [S]
- What do I know? What do I not know? [K]

<table>
<thead>
<tr>
<th>[A] What is the question asking for?</th>
<th>[C] What is/are the concept(s) required?</th>
</tr>
</thead>
<tbody>
<tr>
<td>[I] What is/are the given information/condition(s)?</td>
<td>[S] What is/are the skill(s) required?</td>
</tr>
</tbody>
</table>

Figure 2. The ACISK framework.

This framework encapsulates the four phases of problem solving identified by Polya (1945): (1) Understand the problem, (2) Devise a plan, (3) Carry out the plan, and (4) Look back. Similar to Polya’s problem solving model, learners have to first understand the problem by asking themselves “What is the question asking for?” and “What is/are the given information/condition(s)?”. They then devise a plan by asking themselves “What is/are the concept(s) required?” and “What is/are the skill(s) required?” before next carrying out their plan. In the process of devising and carrying out their plan, they have to identify and bridge their learning gaps by asking themselves “What do I know?” and “What do I not know?”. Finally, they have to look back and examine the solution obtained and ask themselves again “What do I know?” and “What do I not know?”.

An important element in empowering mathematics learners is the emphasis on metacognitive strategies organized through the ACISK framework. Too often students answer questions procedurally, without really thinking through the knowledge involved and asking themselves
how they might access that knowledge. Schoenfeld (1992) asserts that students’ problem solving skills “can be learned as a result of explicit instruction that focuses on metacognitive aspects of mathematical thinking” and “that instruction takes the form of coaching, with active interventions as students work on problems”. Garofalo and Lester (1985) have also proposed a cognitive-metacognitive framework that, incorporating Polya’s ideas, consists of four categories of activities that are involved in problem solving, namely, orientation, organization, execution, and verification, specifying “key points where metacognitive decisions are likely to influence cognitive actions” (p. 171).

3 Self-Questioning Using the ACISK Framework

In this section, we examine how mathematics learners, using the ACISK framework, may regulate their thought processes so that it becomes a habit of the mind which is necessary to develop self-directed learners.

For example, consider the following problem.

Using the ACISK framework, the mathematics learner may approach and analyse the problem through a series of self-questioning prompts:

(1) What is the question asking for?
   - Find the value of unknown constant \( k \).
   - Find the remainder when \( f(x) \) is divided by \( x + 4 \).

(2) What is/are the concept(s) required?
   - Remainder when divided by linear factor is a constant.
   - Remainder Theorem.

(3) What is/are the given information/condition(s)?
   - Roots of \( f(x) = 0 \).
   - Remainder when \( f(x) \) is divided by \( x - 4 \).
(4) What is/are the skill(s) required?
- Form an expression for \( f(x) \).
- Form an equation in \( k \) and solve it.
- Use Remainder Theorem.

(5) What do I know?
- Remainder Theorem.

(6) What do I not know?
- Form an expression for \( f(x) \).

Through the process of self-questioning and analysis, as shown above, the learner is able to identify the gaps in his/her learning. For example, though he/she may know how to use the Remainder Theorem, he/she may be unable to make use of the information of “the roots of \( f(x) = 0 \)” to “form an expression for \( f(x) \)”.

This framework has been actualised by the authors in mathematics lessons. Problem 1, shown in Figure 3, was given to a group of 30 students of low ability in mathematics and they were asked to use the ACISK framework to analyse how to approach the problem.

The function \( f(x) = x^3 + ax^2 + bx + 4 \), where \( a \) and \( b \) are constants, is exactly divisible by \( x - 2 \). Given that \( f(x) \) leaves a remainder of \(-3 \) when divided by \( x + 1 \),

(i) find the value of \( a \) and of \( b \),

(ii) express \( f(x) \) in the form \((x-2)(x-1-\sqrt{d})(x-1+\sqrt{d})\), where \( d \) is an integer.

*Figure 3. Problem 1*
Figure 4 shows how student A used the ACISK framework to analyse Problem 1.

<table>
<thead>
<tr>
<th>What is the question asking for?</th>
<th>What is the concept(s) required?</th>
</tr>
</thead>
<tbody>
<tr>
<td>i) unknown variables</td>
<td>remainder &amp; factor theorems</td>
</tr>
<tr>
<td>ii) express as product of factors</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>What is the given condition(s)?</th>
<th>What is the skill(s) required?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(2) = 0 )</td>
<td>factorising</td>
</tr>
<tr>
<td>( f(-1) = -3 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>What do I know?</th>
<th>What do I not know?</th>
</tr>
</thead>
<tbody>
<tr>
<td>binding the word at ( a ) &amp; ( b )</td>
<td>express ( f(x) ) in the form ( (x-2)(x+1)(x-3) ) ( \frac{1}{x-1+3} )</td>
</tr>
</tbody>
</table>

Figure 4. Student A’s analysis of Problem 1 using the ACISK framework

It is apparent from Figure 4 that Student A was able to apply the concept of remainder and factor theorems to the given conditions “exactly divisible by \( x - 2 \)” and “leaves a remainder of \(-3\) when divided by \( x + 1 \)” to form two simultaneous equations, \( f(2) = 0 \) and \( f(-1) = -3 \), and then solve them for the values of \( a \) and \( b \). It is also apparent that though she understood that she was required to factorise the expression \( f(x) \), she was unable to express it in the required form. Thus she was able to identify the concept or skill that she still did not know.
Problem 2, shown in Figure 5, was also given to the same group of 30 students of low ability. They were asked to approach and analyse the problem using the ACISK framework.

A function is defined by the equation \( y = x^3 + 3x \).

(i) Find \( \frac{dy}{dx} \).

(ii) Hence show that \( y \) increases as \( x \) increases for all real values of \( x \).

Figure 5. Problem 2

Figure 6 shows how Student B used the ACISK framework to work on Problem 2 and Figure 7 shows her solution to the problem.

Figure 6. Student B’s analysis of Problem 2 using the ACISK framework
It is apparent from Figure 6 that though Student B appeared to have solved the problem, by asking herself “What is the question asking for?” and looking back at “What do I know?” and “What do I not know?”, she was able to identify that her solution was not correct as it was based on the premise that “if $y$ increases” instead of showing that “$y$ increases as $x$ increases”.

As cited in Ball’s study (1990), many students learn mathematics without understanding the underlying principles but merely by the rote process, and Morris (1981) pointed out that “the students who rely on learning by rote and respond to problems with an automatic algorithm will reach a point when their type of learning falls apart”. Students who respond to problems with procedures learnt by rote would probably be paralyzed in their thinking when they encounter problems that are worded or presented differently from what they have memorized. Thus it is important to “teach students to fish”, a characteristic of self-directed learners, than to “give students the fish” which makes them merely rote learners. Following from the descriptions and examples of the use of the ACISK framework, a way to help students to be self-directed learners is to show them a thinking protocol that they can use to solve a mathematics problem, routine or non-routine.
While the polynomials and calculus questions cited above may be somewhat routine, in that they are the style of questions commonly found in mathematics textbooks, these questions can be quite challenging for the mathematics learners of low ability and they are able to provide an opportunity for students to put the ACISK framework into practice before the self-questioning and metacognitive behaviour becomes part of their nature or unconscious assimilation. The use of the framework may also inhibit students’ impulsive responses to problems but support their development of knowledge through metacognitive experiences, so that they will not be “more concerned with the mechanics of solution execution and the tyranny of time than with planning, monitoring and verification strategies” (Stillman, 1993).

4 Reflecting and Identifying Gaps in Knowledge Using the ACISK Framework

In this section, we show how students can also use the ACISK framework as a self-assessment tool for learning after a test or an examination to reflect and identify gaps in their knowledge. It is common that after a test or an examination, teachers would go through the correct answers with the students in class. However, students often ended up being busy with copying of answers instead of identifying where they had done wrongly and what the gaps in their knowledge were.

The authors, instead of showing the correct solutions of test items, gave their students model answers to compare with their answers to test items. The students were given the ACISK framework to ask themselves questions and answer them. The questions were as follows:

(A) Understanding the question
- Did you understand what the question is asking for?
- What did you mistake it is asking for?
- What led you to understand the question wrongly?

(C) Knowing the concept
- Did you know what concept was tested?
ACISK Framework – A Tool for Empowering Mathematics Learners

- Did you use a wrong concept?
- Did you use the concept wrongly?
- What led you to use the wrong concept or the concept wrongly?

(I) Using the information/condition(s)
- Did you use the given information/condition(s) correctly?
- Did you make use of every given information/condition?
- Was there any information/condition that you did not understand/use? If yes, why?

(S) Applying the skill(s)
- Did you know what skill(s) to use?
- Did you apply the skill(s) correctly?
- What led you to apply the wrong skill or the skill wrongly?
- Did you make any careless mistake?
- What led you to that carelessness? Was that careless mistake avoidable?
- Did you have correct presentation?

(K) Knowing what is not known
- What is it that you still do not understand/know?
- What did you learn from this test?
- What would you have done better in the next test?

Figures 8 and 9 show portions of Students C’s and D’s reflection on their incorrect solutions of test items. It is apparent from Figure 8 that for Question 7 Student C was unable to recognise that the equation was a quadratic form (“I did not realise the question was a quadratic equation”) and he did not apply the Zero Product Rule (“even though the RHS had to be 0”) even though he had the concept of the trigonometric identities (“though my concept was right”). For Question 8ii, he did not give his answer in terms of π (“I carelessly left my answers in decimal form, and not in radian form”) when the question had asked for “exact solution(s)”.
As shown in Figure 9, for Question 8(ii) although Student D knew the concept of the domain of angle \( \theta \) ("although I adjust the domain"), he had left out the negative values of \( \theta \) in his answer due to carelessness ("I forgot..."
the (to) generate the negative angles of $2\theta$), and upon reflection, he would write down the condition to remind himself in future (“I should always write the adjustment on the paper like ‘since $-\pi < \theta < \pi$, $-2\pi < 2\theta < 2\pi$’ to remind myself”).

As students reflect on their performance in a test or an examination, they are strengthening their capacity to learn. In her work “Seven Strategies of Assessment for Learning”, Chappuis (2009) presented self-assessment as her fourth strategy and stated that research had confirmed “when students are required to think about their own learning and articulate what they understand and what they still need to learn, achievement improves” (p. 146), and her seventh strategy asked teachers to “provide students opportunities to track, reflect upon and share their learning progress” (p. 269). The ACISK framework thus serves to empower the mathematics learners as it helps to guide their reflection so that they are aware of and can describe their thinking in a way that allows them to identify and close the gap between what they know and what they need to learn.

5 How the ACISK Framework Empowers Students to be Self-Directed Learners

In their work, *How Learning Works*, Ambrose, Bridges, DiPietro, Lovett and Norman (2010) explored how learners learn best and how teachers can appropriately foster their learning. The authors of this chapter affirm that Principles 4 and 7 are aligned to the goals of the ACISK framework as “to develop mastery, students must acquire component skills, practice integrating them, and know when to apply what they have learned” and “to become self-directed learners, students must learn to assess the demands of the task, evaluate their own knowledge and skills, plan their approach, monitor their progress, and adjust their strategies as needed”.

Under Principle 4, Ambrose, Bridges, DiPietro, Lovett and Norman suggested strategies that (1) expose and reinforce component skills, such as diagnosing weak or missing component skills; and (2) facilitate transfer, such as helping students learn when to apply what they have learned. Parallel to this principle and the suggested strategies, the ACISK
framework is able to help learners to identify the skills and knowledge necessary to solve a complex problem, under the components C – “What is/are the concepts required?” and S – “What is/are the skills required?” and also to identify the skills or knowledge they still lack, under the component K – “What do I know?” and “What do I not know?”, so that they can then practise the necessary skill or acquire the knowledge in relative isolation. As argued by the authors in chapter 4 of the book, it is important that learners not only must “acquire a set of component skills, practice combining and integrating these components to develop greater fluency and automaticity”, they must also “understand the conditions and contexts in which they can apply what they have learned” (p. 120).

Principle 7 suggests that metacognitive skills are critical for students to be effective self-directed learners, who are able to recognize what they already know, identify what they still need to learn, plan an approach to learn, review and refine their learning goals so that they can realistically achieve it, and monitor and adjust their approach along the way (p. 191). In the book, the authors suggest that students do not necessarily develop metacognition on their own and that teachers play a critical role in helping students develop the metacognitive skills: assessing the task at hand, evaluating one’s own strengths and weaknesses, planning, monitoring performance along the way, and reflecting on one’s overall success (p. 215). Parallel to this principle and the suggested strategies, the component K in the ACISK framework helps learners to recognize what they already know (“What do I know?”) and also identify what they still need to learn (“What do I not know?”).

With the ACISK framework, learners can also assess their own understanding of subject matter by analysing the patterns of mistakes they make in their work and examinations, and the information they gain from these ongoing analyses can help them identify the concepts or skills yet to learn, adjust their learning strategies, and improve the next iteration of their learning process.
6 Implications to Classroom Teachers on Using the ACISK Framework

The ACISK framework may be used to serve two purposes: (1) to motivate students to self-direct their learning during mathematics instruction, and (2) to help students in their process of problem solving, even when given non-routine mathematical tasks. In this chapter, the ACISK framework is proposed as a tool that may be used to empower our mathematics learners of all abilities, to be self-directed. It may be especially beneficial to learners of low ability as the framework serves as a tool to initiate and organise their thinking processes, indirectly boosting their confidence in doing mathematics. Over a period of time, students will internalise the process even without the framework, and develop the thinking habit and the disposition to problem solving.

To use this framework, it is necessary that teachers in our daily teaching identify what concepts and what skills are, and also purposefully construct learning experiences that help learners pick up mathematics academic vocabulary such as “show”, “find”, “explain”, “factorise”, “solve”, “evaluate”, etc.

Teachers also need to model learning strategies so that learners develop the ability to use these strategies on their own. Teachers must show learners how we ourselves would approach a problem and walk them through the various phases of thinking and analysis. The ACISK framework serves as a suitable guide for that. Let learners hear us “talk out loud” as we assess the task (“What is the question asking for?” and “What is/are the given information/condition(s)?”), assess our knowledge (“What is/are the concepts required?” and “What is/are the skills required?”), and assess our own strengths and weaknesses (“What do I know?” and “What do I not know?”).

7 Conclusion

In this chapter, the ACISK framework is proposed as a tool that may be used to empower our mathematics learners to be self-directed. The learning process, of course, does not end with just answering the six questions in the framework. Teachers need to work with learners on the
actions required to close the identified gap between what they know and what they need to learn, and also to constantly reassess and adjust as they progress. Teachers also need to scaffold learners in their metacognitive processes and the ACISK framework may be used as a cognitive support that teachers provide learners with in their early stage of learning before gradually removing it as learners develop greater mastery.

The ACISK framework does not automatically make mathematics learners self-directed. Teachers need to complement it with other strategies, such as giving constructive feedback, to teach students to self-regulate and self-manage their learning. Grow (1991) highlighted that “good teaching matches the learner’s stage of self-direction and helps the learner advance towards greater self-direction” (p. 125).

References


Chapter 16

Empowering Students through Inquiry

Steve THORNTON

There is a considerable literature about the importance of inquiry in school mathematics. Yet there seems relatively little agreement about what qualifies as inquiry in school mathematics, with approaches ranging from highly engineered tasks with clear learning intentions to very open, student-led inquiries where the outcomes are unclear. Whatever our conception of inquiry, the goal must be to encourage students to wonder, to ask questions, to seek patterns and explain solutions. Such a goal empowers students to both make sense of the world and to begin to work like mathematicians. This chapter describes the spirit of inquiry in the reSolve: Mathematics by Inquiry project, an Australian Government funded project designed to promote inquiry approaches to mathematics from Kindergarten to year 10. The approach adopted empowers students to reason mathematically, emphasizing the key aspects of formulating complex problems and communicating and evaluating the solution. The project is underpinned by the reSolve: Mathematics by Inquiry Protocol, a framework used to inform the development of all materials in the project and to provide teachers with a robust theoretical and practical framework for teaching inquiry-oriented mathematics lessons.

1 A Rationale for Inquiry

Curriculum documents across the world emphasise the importance of students developing curious and questioning approaches in mathematics.
For example, the Singapore Primary Mathematics Syllabus (Singapore Ministry of Education, 2012) states: “to encourage students to be inquisitive, the learning experiences must include opportunities where students discover mathematical results on their own” (p. 20). It goes on to describe an environment of teacher-directed inquiry, where students are led to “explore, investigate and find answers” (p. 24). In the approach recommended in the Singapore syllabus, inquiry follows activity-based learning that provides a common problem solving experience, and is, in turn, followed by a period of direct instruction in which teachers “draw connections, pose questions, emphasise key concepts, and role model thinking” (p. 24).

Although not using the term inquiry explicitly, the Australian Curriculum: Mathematics, (Australian Curriculum and Assessment Reporting Authority [ACARA], 2013) emphasizes the importance of students reasoning about mathematics and solving problems. A key aspect of this is developing the general capability of critical and creative thinking through “generat(ing) and evaluat(ing) knowledge, ideas and possibilities, and us(ing) them when seeking solutions” (p. 2).

The PISA 2006 Mathematics Framework emphasizes the importance of students being able to make “the well-founded judgements and decisions needed by constructive, engaged and reflective citizens” (Stacey & Turner, 2014, p. 5). It views students as active problem solvers, who are able to formulate situations mathematically, employ mathematical concepts, facts, procedures and reasoning, and interpret, evaluate and apply mathematical outcomes.

All of the above represent a growing emphasis on mathematics as a process rather than a product. The various documents see students as active learners, asking questions and generating knowledge in order to solve problems that are meaningful to them. They see teachers as role models and facilitators of this process, taking an active role in stimulating students to develop conceptual understanding, procedural skills and the capacity to reason mathematically. Students develop productive attitudes and dispositions, as well as the metacognitive skills needed to monitor their own learning (Singapore Ministry of Education, 2012).
In this largely theoretical chapter, I will argue that an inquiry orientation to mathematics is an effective way to empower students to develop both the cognitive and dispositional outcomes described in these documents. I commence with an overview of different conceptions of inquiry, and argue that, rather than seeing inquiry as a noun, we ought to consider it as a verb, representing the process of seeking after knowledge through posing problems and being open to ideas. I then discuss the evidence for an inquiry approach, arguing that there are both cognitive and affective benefits for students. Following this, I discuss the approach adopted by the reSolve: Mathematics by Inquiry curriculum and resources project in Australia, describing in particular the Protocol that contains the vision for the approaches being developed. I give two brief examples from the draft resources, one a classroom activity written for students in the middle years of school, and the other a professional resource designed to orient teachers to some strategies they might use to sustain challenge in an inquiry-oriented classroom.

2 Conceptions of Inquiry in School Mathematics

The notion of inquiry as both a process and goal of education dates back to Dewey (1916) and his concern for having education serve the cause of democracy. Dewey highlighted what he termed interest and discipline, describing interest as active involvement in something that will make a difference, and discipline as the power to persist by using the resources at hand. He saw these two concerns as closely connected, not opposed, despite the recognition that a form of discipline is often employed to encourage persistence on tasks that hold little interest. In fact he saw the imposition of external rewards or punishments as antithetical to good education, describing the act of securing attention by bribes of pleasure or imposition of sanctions as a “‘soup-kitchen’ theory of education” (p. 132).

Dewey considered that for something to be a genuine object of study, “that is, of inquiry and reflection” (p. 140) it needed to be something whose outcome affects the person engaged in the activity.
Numbers are not objects of study just because they are numbers already constituting a branch of learning called mathematics, but because they represent qualities and relations of the world in which our action goes on, because they are factors upon which the accomplishment of our purposes depend (Dewey, 1916, p. 140).

Dewey therefore argued strongly against the common practices of drill and practice exercises designed to increase efficiency, and even against an accumulation of knowledge as an end in itself. He saw both knowledge and procedural fluency as a means to an end—that of intelligent engagement in purposeful activities that made a difference to people’s lives. In this way education would serve the cause of democracy by empowering people to act in the world, rather than reproducing the status quo.

Dewey’s notion of inquiry serving the cause of democracy echoes today, particularly in an environment where the quality of educational systems, at least in the USA but arguably internationally, is measured in “uber-mechanical, fixed-aim-oriented accountability systems such as NCLB and its various state-level standards and high-stakes testing regimes” (Stemhagen & Smith, 2008). Stemhagen and Smith argue that mathematics may be the last bastion of curricular certainty and that inquiry for democratic action, as described by Dewey, is conspicuously lacking from most contemporary mathematics teaching. They give an example of how students might engage in a substantial inquiry into fair methods of scoring in areas such as educational assessment and sports rankings. They claim that such an activity engages students in thinking about issues that matter to them, in the sense that Dewey used the term “interest” (Dewey, 1916, p.130).

Although not as obviously related to democratic ideals, Makar (2012) uses the example of “what is the best brand of bubble gum?” To answer the problem students need to debate what is meant by “best”. This might be a matter of taste, the size of a bubble that can be blown, or the length of time the bubble lasts. Students then need to find ways of measuring these attributes, make decisions about how accurately and often to measure, organize and analyse data collected and make and justify conclusions. Makar terms these four phases of inquiry discover, devise,
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develop and defend. She concludes that for such inquiries to lead to significant learning for students, there needs to be multiple opportunities with appropriate support and scaffolding from the teacher.

Inquiries such as these require students to spend considerable time in phases of inquiry such as exploring situations, planning investigations, experimenting systematically, interpreting and evaluating, and communicating results (Artigue & Blomhøj, 2013). In carrying out such inquiries students:

- create their own scientifically and mathematically oriented questions;
- give priority to evidence in responding to questions;
- formulate explanations from evidence;
- connect explanations to scientific and mathematical knowledge; and
- communicate and justify explanations (National Research Council Mathematics Learning Study Committee, 2001, p.27).

There are many other examples of similar inquiries, described in European projects such as PRIMAS and Fibonacci (Maaß & Artigue, 2013), in web-based materials such as brilliant.org (Brilliant, 2016), and in assessment and professional learning resources such as Bowland Maths (Bowland Charitable Trust, 2007-2014).

However, this form of inquiry, which I term Inquiry (with a capital I) is not the only way in which students might participate in an inquiry process. The above examples show situations in which students ask and answer questions about contexts in which mathematics plays a central role. Yet even the most curriculum-focused lesson can become an opportunity for inquiry given an appropriate pedagogical approach. Artigue & Blomhøj (2013, p. 797) define inquiry-based pedagogy loosely as “a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work”. I term this inquiry (with a lower case i) in which inquiry is seen not so much as a product as a process (Calleja, 2016). Such a process of inquiry allows students to inquire into mathematics, as well as inquiring with mathematics (Staples, 2007).

I argue that, in order to empower students to think as mathematicians, an orientation to inquiry should therefore underpin all learning in mathematics, in that students should be encouraged to ask questions, test
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ideas, seek meaning and explain ideas throughout their mathematics learning. Such a spirit of inquiry is essential to producing mathematics, if not in the way that professional mathematicians produce mathematics, at least as a student may produce it, and is stimulated by prompts such as “ponderings, what ifs, seems to be that’s, and it feels as thoughs” (Burton, 1999, p. 30). Such an orientation to inquiry is essential for mathematical empowerment, as it positions students as builders rather than receivers of mathematical knowledge.

As discussed in Section 4, these and other aspects of pedagogical approaches that underpin an inquiry-oriented approach to school mathematics have been incorporated into the reSolve: Mathematics by Inquiry Teaching Protocol.

3 Evidence for the Benefits of an Inquiry Approach

3.1 Cognitive benefits

The research evidence supporting inquiry-based learning generally, and in mathematics in particular, is contested. Hattie (2008), for example, found an effect size of 0.31 for inquiry learning, which he claims is relatively small. He argues that this is often because students do not have the requisite knowledge to conduct meaningful inquiries, and that it is introduced too early. Hattie did note, however that inquiry-based teaching had significant effects on critical thinking processes, and was shown to “produce transferable thinking skills as well as significant domain benefits, improved achievement, and improved attitude towards the subject” (p. 210).

The recently released results of the 2015 Programme for International Student Assessment (PISA) similarly finds that teacher-directed approaches to science are more effective than inquiry-based approaches, as these are defined by the PISA questions (OECD, 2016). In almost all countries participating in PISA an emphasis on teacher-directed strategies such as explaining scientific ideas and leading class discussions was positively associated with higher scores on the PISA assessment. On the other hand, greater exposure to inquiry-based instruction, including
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pedagogical strategies such as doing practical experiments or arguing about ideas, was negatively correlated with achievement. It would be unsurprising if similar findings were not obtained in school mathematics. As I discuss below, however, this positions teacher direction and inquiry as opposite and incompatible extremes rather than as complementary and mutually supportive approaches to teaching and learning.

Hattie’s somewhat lukewarm endorsement of inquiry finding is echoed by Kirschner, Sweller and Clark (2006) who argue that in almost all situations highly guided instruction is more effective than minimally guided instruction, and that this advantage only begins to recede when learners have a high level of prior content knowledge. Kirschner et al. define learning as a change in long-term memory, and argue that the load on working memory imposed by approaches such as inquiry learning and problem-based learning works against effective learning by requiring the learner to search a problem space for relevant information. They argue that the working memory thus becomes unavailable for learning. Kirschner et al. claim that advocates of inquiry learning have confused epistemological and pedagogical aspects of the discipline, assuming that the way mathematicians and scientists operate in the world ought to be mirrored in the way students learn. This confusion leads to the treatment of novices as experts, which may set them up for failure and therefore be detrimental to learning. They cite empirical evidence supporting strongly guided instructional practices, such as the use of worked examples over minimally guided instruction, which they argue is characteristic of inquiry-based learning.

In a strong critique of Kirschner et al.’s argument, Hmelo-Silver, Duncan and Chinn (2007) claim that they have misrepresented inquiry learning and problem-based learning. Hmelo-Silver et al. present several examples of guidance and scaffolding that underpins many examples of inquiry learning, claiming that such scaffolding supports students’ learning of both content and disciplinary ways of working. Unlike worked examples, scaffolding an inquiry process may also problematize important aspects of students’ work, focusing them to engage more deeply with key ideas and strategies. Hmelo-Silver et al. present evidence that inquiry-based learning is at least as effective as strongly guided instruction in meeting cognitive goals, but significantly more effective in realizing goals
such as flexible thinking and collaboration that prepare students to be adaptive learners.

Bruder and Prescott (2013) reach similar conclusions. They distinguish between structured inquiry, guided inquiry and open inquiry. They describe structured inquiry as a situation in which the teacher provides both the problem to be solved and the method and materials needed to solve it, guided inquiry as a situation in which the teacher gives the problem and materials, but allows the student to choose an appropriate method, and open inquiry as a situation in which students are required to find both the problem and the method. Like Hmelo-Silver et al., Bruder and Prescott conclude that what is at stake is not so much the effectiveness of inquiry-based learning, but rather what we value in mathematics. They suggest that if we value performance on high stakes tests, then inquiry learning is likely to be less effective than more traditional teacher-directed instruction. However, if we want students to understand mathematics deeply, enjoy mathematics and work through problems to a conclusion, then there is strong evidence for the benefits of inquiry-based learning. That is, if our goal is to empower students as mathematicians rather than to produce high test results, elements of an inquiry approach are essential in school mathematics.

3.2 Affective and social benefits

Evidence for the benefits of informal, inquiry-oriented approaches to school mathematics date back almost a century (Benezet, 1935). Affective and social benefits include greater engagement with mathematics, the capacity to apply mathematics to real world problems, the development of attributes such as persistence and confidence, and greater equity in providing access for all (e.g. Boaler, 1998; Boaler & Greeno, 2000; Lampert, 2001). Engaging in open-ended problem solving develops beliefs about mathematics that more accurately reflect the nature of the discipline than those of students undertaking more traditional forms of instruction (Schoenfeld, 1992).

Calder (2013) describes a situation in which Grade 10 students engaged with an authentic statistical inquiry related to the Olympic Games. He claims that student ownership of the context, the research
question, the inquiry process and the people with whom they worked led to high levels of motivation and engagement. Like the environments described by Hmelo-Silver et al., the learning environment was carefully structured, including needs-based workshops to explicitly teach new content as needed such as box-and-whisker plots. Calder highlights the importance of the teacher making both the content and the mathematical processes explicit, and to emphasise connections between the mathematical content and other curriculum areas.

Closely related to the importance of ownership is what DeBellis and Goldin (2006) term meta-affect. This is what enables people, in the right circumstances, to experience emotions such as fear or frustration as pleasurable, or as a spur to further problem solving. Two particular aspects of affect highlighted by DeBellis and Goldin are mathematical intimacy and mathematical integrity. Mathematical intimacy involves emotions such as warmth, excitement, amusement or aesthetic appreciation accompanying understanding. Mathematical integrity is the individual’s response when a solution is “right”. It is a commitment to truth and understanding. DeBellis and Goldin suggest that students with strong mathematical integrity and intimacy have the potential to engage in powerful problem solving. In turn, mathematical integrity and intimacy are much more likely to develop when students have opportunities to engage in the processes of mathematical inquiry.

Both mathematical intimacy and integrity are enhanced when students develop an aesthetic appreciation for mathematics and problem solving (Sinclair, 2004). Sinclair argues that the aesthetic plays a critical role in the inquiry process, and identifies three important roles: the evaluative, the generative, and the motivational. The evaluative role is the most widely recognised—it is that sense of beauty and elegance when one sees a result or proof in mathematics that is simple, compelling, insightful, deep, general or important. The generative aesthetic as an expectation of order that “guides the actions and choices that mathematicians make as they try to make sense of objects and relations” (p. 270). The generative aesthetic involves playing, in which the mathematician/student tries to make connections and look for patterns or appealing structures; establishing intimacy, in which the mathematician/student develops ownership and finds a way of describing and naming unknown territory; and capitalizing


on intuition, in which the mathematician/student senses the rightness of a result or line of argument. Finally, the motivational role of the aesthetic is a necessary precursor to mathematical inquiry; it is the sense of surprise, delight or paradox that can prompt the mathematician/student to engage in extended struggle to understand new ideas or to resolve tension or curiosity.

It is clear, then, that even though evidence for cognitive benefits of an inquiry approach to school mathematics may be debated, there are significant affective benefits. These include the development of stronger beliefs about mathematics, the development of a sense of ownership, greater mathematical intimacy and integrity, and developing a sense of the aesthetic both as a product of mathematics and as a strong motivator for pursuing mathematical inquiry. Developing affective responses to mathematics such as these is essential in empowering students as learners and doers of mathematics.

4 The Australian reSolve: Mathematics by Inquiry Project

reSolve: Mathematics by Inquiry is an Australian Government funded project designed to promote a spirit of inquiry in students from Kindergarten to year 10. The project is managed by the Australian Academy of Science in collaboration with the Australian Association of Mathematics Teachers. The project has two distinct arms: the first is the development of a coherent suite of resources promoting mathematical inquiry; the second is the engagement of the profession. The resources include professional learning resources focusing on important elements of inquiry such as including all students and maintaining challenge, highlighted by exemplary classroom resources addressing key components of the Australian Curriculum: Mathematics (Australian Curriculum and Assessment Reporting Authority [ACARA], 2013). Engagement of the profession occurs through an extensive trialling and feedback process, and ultimately the recruitment of 240 Champions across all states and territories of Australia and from a diverse range of year levels and backgrounds. The project is underpinned by the reSolve: Mathematics by Inquiry Protocol, described below.
4.1 The reSolve: Mathematics by Inquiry Protocol

The reSolve: Mathematics by Inquiry Protocol articulates those elements of the mathematics, the tasks and the learning environment that we believe will promote a spirit of inquiry and enable students to successfully gain the cognitive and affective benefits discussed in the literature. The Protocol is based loosely on the Teaching for Robust Understanding (TRUMath) dimensions (Schoenfeld, Floden, & the Algebra Teaching and Mathematics Assessment Project, 2014). The three key elements in the Protocol are described as:

- reSolve mathematics is *purposeful*
- reSolve tasks are *challenging yet accessible*
- reSolve learning environments promote a *supportive, knowledge-building* culture

By mathematics that is purposeful we wish to challenge perceptions that mathematics is nothing more than a body of disconnected facts or procedures described in a curriculum document. We seek to highlight connections between mathematical ideas and between mathematics and the real world by focusing on important mathematical ideas that give students power in their lives. We seek to acknowledge mathematics as a creative and imaginative endeavour, continually changing and developing in a technological society.

By tasks that are challenging yet accessible we wish to challenge perceptions that mathematics is for the few, and assert that it ought to be both challenging and accessible for all students. reSolve tasks thus activate existing knowledge and develop new knowledge, and students explore relationships between key ideas by working on meaningful tasks. They engage students in sustained inquiry, problem solving, decision making and communication using structured tasks and technologies to optimise students’ mathematical development. They use evidence of students’ progress to inform feedback and subsequent teaching action and provide prompts and activities that meet a range of student capabilities.

By supportive, knowledge-building environments we wish to challenge a view that mathematics is best learnt through demonstration, reproduction and repetition. We seek to promote environments that sustain
higher order thinking through the active role of both teachers and students and build success through collaborative inquiry, action and reflection. We seek to challenge existing student ideas or misconceptions and use mistakes as opportunities for learning. We seek to build positive dispositions such as productive struggle and the confidence to take risks.

Taken together the project team believes that these three key aspects of mathematics, tasks and environments can create an inquiry orientation to mathematics that is relevant, empowering and that lead to strong cognitive and affective outcomes. In the next section, I give two brief examples from the classroom resources and show how they highlight these three aspects of the Protocol.

4.2 The reSolve: Mathematics by Inquiry Sums of Squares lesson

The Sums of Squares task explores the hypothesis of Diophantus, an ancient Greek mathematician, that any positive integer can be represented as the sum of four square numbers. The task has been trialled several times, with groups of Year 7 or 8 students of varying abilities. From a curriculum perspective, the task is essentially a fluency exercise in recognizing square numbers and in finding efficient ways to combine perfect squares to produce other numbers. More importantly though, it aims to promote an attitude of curiosity among students, in that they ask questions about why some types of numbers require four squares, while others only require three or two. In this way, it attempts to mirror how a mathematician might pose questions, seek generalisations, and explain results.

The task commences with no initial explanation, but rather with a whole-class demonstration that \(46 = 36 + 4 + 1 + 1 + 1 + 1 + 1\). By then exchanging four 1s for a 4, the equation \(46 = 36 + 4 + 4 + 1 + 1\) is shown to students. Rather than telling students the purpose of the activity, we leave students to wonder about the process and what it is intended to demonstrate. After a short time, the number 9 is used to replace \(4 + 4 + 1\), resulting in \(46 = 36 + 9 + 1\). At this stage, the students realise that perfect squares are being summed to produce other numbers, and they are asked whether three squares is the minimum number required to make 46. This prompts the inquiry question “What is the minimum number of perfect squares that need to be summed to produce any number?”
We do not initially state the hypothesis, but instead invite students to try to make all positive whole numbers between 1 and 120 as the sum of square numbers. This is a collaborative exercise in which we invite students to show their results using sticky post-it notes on a class display (Figure 1). The display consists of 15 rows of eight columns headed with the numbers 1 through 8, continuing with 9 through 16, and so on up to 113 to 120. This is a key design feature of the task as it highlights patterns and regularities and encourages students to make hypotheses such as that numbers of the form 3, 11, 17, … (column 3) all require three squares.

After some time we ask students to make a conjecture about the minimum number of squares required for any number. The class display prompts the conjecture that all numbers can be written as the sum of no more than four squares, but there are usually several numbers for which the students have used five or more. This prompts a search for more efficient representations. It is only after students have independently suggested and tested their conjecture that we introduce the hypothesis of Diophantus that all positive whole numbers are the sum of not more than four square numbers (or exactly four if we allow 0). We also explain that the result was proved more than one thousand years later by Lagrange.

In our classroom-based trialling, we found that, after an initial period of wondering, students quickly grasped what the task was about. Some asked about the purpose of the activity and how it related to other work they had been doing in the normal course of their school mathematics, but most were interested in trying to find the smallest number of squares because it was an intriguing problem to which the answer was not obvious. We did, however, take care to highlight the historical aspects of the problem, emphasising that mathematics is purposeful; it is a living, growing human endeavour.

We found that students engaged deeply in the pursuit of mathematical knowledge, very much in the way that a mathematician might. Students found the task both challenging and accessible. It was accessible as students were quickly able to express many numbers as being themselves perfect squares, and to make numbers that are one or four more than a perfect square by using one additional square number. However, other numbers were less obvious, and students initially used five or more squares to make them. For example, it was common for students to make
71 as $64 + 4 + 1 + 1 + 1$. This presented a challenge, as it appears to contradict the hypothesis that at most four squares are needed. We found that the use of post-it notes created a knowledge-building culture as students were able to look at the results on the board and both self-correct and build on and try to improve on what others have posted. In the case of 71, one group of students worked together to write it as $49 + 9 + 9 + 4$, and replaced the original post-it note with this more efficient representation.

For those students who have some knowledge of modulo arithmetic it is possible to explain why numbers of a certain form require three or four squares, even though a proof that four squares is sufficient may be beyond high school students. The task is thus inherently challenging and accessible, while the use of the class display board and post-it notes leads to powerful correction and improvement, important aspects of a supportive knowledge-building culture. An obvious extension is to examine positive whole numbers as the sum of cubes, which students can explore using technology.

The sums of squares task thus illustrates mathematics that is purposeful, challenging and accessible, and that promotes a collaborative, knowledge-building culture. It is an inquiry into mathematics itself, one in which students pose questions, seek patterns and try to generalise their results. In the course of their investigation into the minimum number of squares required to write any number, students work like mathematicians, actively building their knowledge through self-correction and refining the results of others.
4.3 A reSolve: Mathematics by Inquiry professional resource

The classroom resources, such as the sums of squares task described above, are intended to be exemplary resources that illustrate important pedagogical strategies to effectively implement an inquiry approach to school mathematics. These strategies are described in a set of professional resources, built around the reSolve: Mathematics by Inquiry Protocol, that are designed to be used by groups of teachers wishing to improve their practice. One such professional learning module focuses on initiating and sustaining challenge.

Teachers who undertake the professional learning in this module commence by considering three approaches to teaching: one in which the teacher simply poses a problem with no student direction; a second in which the same problem is posed with teacher guidance but without explicit instruction in the techniques required to solve the problem; and a third in which the same problem is posed again with explicit instruction of
the techniques required. They discuss the affordances and constraints of each approach, not suggesting that one is better than another, but rather that each has its place in a well-balanced programme, and that each offers challenge of a different nature. The introduction to the module thus gives teachers a practical task to both engage in the professional learning and situate their future deliberations.

The module then reviews some of the research literature relating to sustaining challenge in an inquiry environment (e.g. Anthony & Walshaw, 2009; Christiansen & Walther, 1986; Hiebert & Grouws, 2007). Teachers are asked to consider several quotations and to discuss the extent to which each of the following aspects of sustaining challenge resonates with them and their experience:

- Starting with a challenge can build a sense of success;
- The type of task is important;
- High level thinking enhances learning;
- Cognitive activation builds positive motivation;
- Challenge needs to be appropriate and productive;
- Challenge is beneficial for all students.

This sets the scene for an in-depth discussion of performance and growth mindsets (Dweck, 2006).

We see the development of a growth mindset as a critical aspect of students’ adoption of a spirit of inquiry in mathematics. Students who have a growth mindset tend to have a resilient response to failure, remain focused on mastering skills and knowledge even when challenged, do not see failure as an indictment on themselves, and believe that effort leads to success (Dweck, 2006). Crucially the professional learning module argues that teachers can help to change a student’s mindset from one that is focussed on performance where pleasing the teacher is paramount to one that is focussed on growth with inquiring into mathematics at the centre. Having a growth mindset is thus critical in empowering students.

After further exploration of some task examples from the reSolve classroom resources, we ask teachers to reflect upon their own practice. Specifically we ask teachers to consider and discuss the following questions:

- In your current teaching, how long do you usually let students struggle before you “rescue” them?
• In your current teaching approaches, what is the balance between inquiry approaches and teacher direction?
• What is something new that you will try in your class because of this module?

Asking teachers to reflect upon and discuss their own practice empowers them to make changes that we hope will in turn, empower students. In this way the professional and classroom resources work together to present a spirit of inquiry that empowers students both cognitively and affectively.

5 Conclusion

This chapter has described the spirit of inquiry in the reSolve: Mathematics by Inquiry project, an Australian Government funded project designed to promote inquiry approaches to mathematics from Kindergarten to year 10. The approach adopted empowers students to reason mathematically, emphasizing the key aspects of formulating complex problems and communicating and evaluating the solution. I have described the reSolve: Mathematics by Inquiry Protocol, a framework used to inform the development of all materials in the project and to provide teachers with a robust theoretical and practical framework for teaching inquiry-oriented mathematics lessons. Whatever our conception of inquiry, the goal is to “allow…students to wonder why things are, to inquire, to search for solutions, and to resolve incongruities” (Hiebert et al., 1996, p.14). Such a goal empowers students to both make sense of the world and to begin to work like mathematicians.
References


Chapter 17

Developing Self-Regulated Learners in the Primary Mathematics Classroom

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Self-regulation is an essential characteristic of effective learning. Hence, teachers are empowering students in the learning of mathematics when they develop self-regulated learners in the primary mathematics classrooms. There are many self-regulated learning models in the literature. In this chapter, we describe the phases and processes of self-regulation adapted from a researched model and highlight some instructional approaches that teachers can use in the primary mathematics classroom to cultivate self-regulated learning behaviours.

1 Introduction

“The 21st century is often characterised as one where the world we live in is complex, highly interconnected and rapidly changing” (Toh & Kaur, 2016, p. 2). In Singapore, the framework for 21st century competencies and student outcomes was introduced by the Ministry of Education to help students deal and thrive with the challenging demands of 21st century life. Critical for students is an adaptive capability to respond and function well in this world, in particular, the development of self-regulated learning for students to control, monitor and evaluate their learning. Self-regulation is an essential characteristic of effective learning in mathematics of learning (Pape, Bell & Yetkin, 2003) and is part of metacognition in the Singapore mathematics framework, a key feature of the Singapore mathematics curriculum.
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2 Phases and Processes of Self-Regulated Learning

Self-regulated learning (SRL), or self-regulation is “an active, constructive process whereby learners set goals for their learning and then attempt to monitor, regulate, and control their cognition, motivation, and behaviour, guided and constrained by their goals and the contextual features in the environment” (Pintrich, 2000, p. 453). This definition is similar to other models (e.g. Butler & Winne, 1995; Zimmerman, 1989, 2000) of self-regulated learning. In fact, Butler and Winne (1995) defined the self-regulation learning cycle as the cyclical process of setting goals for greater learning, choosing appropriate strategies to work towards those goals, evolving the strategy during the implementation phase, monitoring the effects of their strategies and then recalibrating accordingly. The self-regulated cycle gives students a sense of personal control, which has been shown to be a major source of intrinsic motivation to continue own learning (Zimmerman, Bonner & Kovach, 1996; Dweck, 2000). Self-regulated students focus on how they activate, adjust and follow-through specific learning practices in order to succeed (Zimmerman, 2002).

According to Winters, Greene and Costich (2008, p. 430), the following common assumptions were identified by Pintrich (2000) and Zimmerman (2001) among the many SRL models in the literature: Learners are

1) Active in constructing their own meanings and goals from the various influences in their surrounding environment & their own internal cognitive systems

2) Capable of monitoring and controlling the cognitive, motivational, behavioural, and contextual aspects of learning

3) Capable of setting goals for their learning and it is against these goals that learning is monitored, with control of learning processes influenced by the results of this evaluation.

4) Regulation can be constrained or facilitated by intra-individual factors (e.g. biology & development) as well as extra-individual influences such as context.

The key assumptions of SRL of goal-setting, self-control, monitoring and evaluation can be seen in the problem-solving process flowchart in Teaching Primary School Mathematics: A Resource Book for primary
mathematics teachers (Foong, 2009, p. 80). The flowchart is an adapted version of Pólya’s model (Pólya, 1971) and illustrates an example of how learners can self-regulate their mathematical problem solving process. Another version of the flowchart was also in the Primary Mathematics curriculum document (Ministry of Education, 1990) to guide teachers in problem solving instruction. Lee, Yeo and Hong (2014) examined the impact of using a metacognitive scheme that focuses on the understanding and planning stages of Pólya’s four-stage approach to examine the role of metacognition in self-regulated problem solving. Their studies reported that the metacognitive-based scheme enhances students’ problem solving and developed “productive habits of mind” (p. 475). Polya’s four steps to problem solving was also used to help primary 6 students monitor their strategy implementation in a study by Teong (2011). This strategy was reported to be useful in helping students self-regulate their problem solving process as they become conscious of their strategy implementation when solving problems.

In the following sections, we discuss and provide examples to illustrate the phases and processes of self-regulation of learning situations specific to the mathematics classroom. The phases and processes of self-regulation in this chapter is adapted from Zimmerman (2000) and Leidinger and Perels (2012). There are three phases when the learner faces a learning situation, namely, pre-action phase, action phase and post-action phase.

Figure 1 shows the three phases. The terms pre-action, action and post-action are from Schmitz and Wiese (2006) to differentiate the three phases during the learning process.
2.1 Pre-action phase: Set goals

Set goals. This is the beginning of the self-regulation process which we perceive as the “Where are we going?” (Bransford, Brown & Cocking, 2000) component of self-regulated learning. Here, the learner set goals with respect to the task such as “deciding upon specific outcomes of learning or performance” (Locke & Latham (1990) cited in Zimmerman, 2000, p. 17). Specific proximal or attainable goals can also be set for upgrading knowledge. Those goals will in turn drive their cognitive engagement (Butler & Winne, 1995). Students who set specific and proximal goals for themselves have shown superior achievement and perceptions of self-efficacy (Zimmerman, 2002).

Goal setting is a very important part of the initial phases of self-regulatory processes. “When students set appropriate goals for their mathematics learning, this can facilitate their self-regulation by enhancing their commitment to attaining them and by providing clear standards against which to monitor their progress (Schunk (2001) cited in Fadlelmula, Cakiroglu, & Sungur, 2015, p.1356). Through the setting of
goals, students will be more exposed to what was meant by specific goals, long term and short term goals and how to set more realistic goals.

Strategic planning and self-motivation beliefs. Engaging students in planning how to reach the goal, that is, strategic planning of students’ learning is also part of the beginning of the self-regulation process. “Self-regulative strategies are purposive personal processes and actions directed at acquiring or displaying skill” (Zimmerman, 2000, p. 17). For example, appropriate methods or strategies can be selected for the task to aid students’ mastery of a skill or to enhance performance. Students’ self-motivation beliefs are important components of self-regulated learning as they initiate the learning process, affect students’ engagement with the learning situation and their performance.

2.2 Action phase: Self-control, monitor and regulate

Self-control. As students embark on the learning situation, to engage them cognitively, they need to be in control of their learning and be able to self-observe and track their learning progress. For students to gain control of their learning, staying focused on the task on hand and getting rid of distractions is important. Self-control can also be achieved when students are able to self-instruct themselves. Self-control technique such as mental pictures is widely used “to assist coding and performance” (Zimmerman, 2000, p. 19). To be in control of a learning situation, knowing and employing strategies to analyze and complete a task is essential. That is, knowing and employing a number of cognitive and metacognitive strategies, particularly elaboration and organizational cognitive strategies (such as, reorganizing and connecting ideas) that support understanding at a deeper and more conceptual level.

Self-observation. Self-observation is necessary for pupils to regulate. This stage is where the students ask “Where am I now?” and they pay deliberate attention to the aspects of their behaviour to sieve out their determinants and effects (Schunk, 1996). In order to capture evidence of progress, the students can be involved in the recording of some aspects of their learning so that they can track “specific personal processes or actions that affect
their learning” and the effects it produces (Zimmerman & Paulsen, 1995, p. 14). Self-recording increases students’ self-awareness, which will produce a readiness essential for personal change in the post-action phase. Furthermore, allowing students to track their own progress, in addition to providing frequent feedback to students’ performance, “are suggested best practices to maintain motivation and develop self-regulated learning” (Rave & Golightly, 2014, p. 539). Students can also engage in personal experimentation to check if changes in the strategy or technique used improve any aspect of their learning and performance.

2.3 Post-action phase: Self-judgement and self-reaction

Self-judgment (Self-evaluation & Causal attribution). Zimmerman describes self-judgment as consisting of two processes: self-evaluation and causal attributions. Self-judgment involves self-evaluation of one’s performance (comparing self-monitored information against a goal) and “attributing causal significance to the results” (Zimmerman, 2000, p. 21), e.g. whether poor performance is due to one’s limited ability or to insufficient effort.

Effective learners self-evaluate by comparing their progress against the task criteria to generate judgments about how they are doing. As the learners evaluate if they have met their learning objectives, they were able to assess their own understanding of, e.g. a mathematics topic, and were constantly directed to the learning objectives or goals. If they find themselves not meeting the goals, they will then adjust their learning activities accordingly. At the same time, feedback is strategically used to diagnose challenges and solve problems (Butler, 2002).

Self-reaction (Self-satisfaction & adaptive / defensive inferences). Satisfaction with one’s learning outcomes is an important part of learning. “When self-satisfaction is made conditional on reaching adopted goals, people give direction to their actions and create self-incentives to persist in their efforts” (Zimmerman, 2000, p. 23). Conclusions are then drawn about the modifications to be made in one’s self-regulatory approach during one’s subsequent efforts to learn or perform. Here, the learner is engaged in adaptive or defensive inferences.
In this phase, students can be asked whether they are satisfied with their performance and learning acquired and thereafter decide how they can make changes to their self-regulatory approach in future efforts to learn or perform. The big question to pose to the students here would be “Are you happy with ……”, “What are you going to do about it (now that you have identified what are the challenges and what works for you)?” For example, if the students reflected that certain strategies help them learn and perform, they may retain those strategies and use those strategies in subsequent learning situations. However, if the students reflected that their poor performance is a result of a strategy that may not be effective to them, they can choose another strategy to achieve their goals.

Proof prompts, adapted from a guidebook on self-assessment and goal setting by Gregory, Cameron and Davis (2000) can be used to help students reflect deeply about their learning processes. Example of prompts are “I could have done better if ….” and “Before my next quiz, I would make the following changes ….”. When students are involved in self-assessment, they engage in self-regulated learning skills such as setting goals, planning strategies to reach those goals, understanding their own strengths and weaknesses. Furthermore, when students self-assess, their learning gaps will be more evident to their teachers and thus helping teachers focus on their students’ learning and thinking processes rather than on merely the end-product (Gregory, Cameron & Davies, 2000).

Self-assessment also supports teachers in that they can see the gaps between what they have taught and what their students have learnt. They will look beyond the product and consider students’ thinking about their learning as part of their formative assessment. In doing so, teachers would then be able to pace their teaching according to their students’ needs, while students consolidate their learning before moving on to the next mathematical concept within the spiral mathematics curriculum. Teachers can also support students’ learning by guiding them on steps they should take to move forward to their learning destinations (Davis, 2007). Learning is thus enhanced when students see their strengths, understand what they need to work on and how to work on them, and are able to set personal learning goals (Gregory, Cameron & Davies, 2000).

Self-assessment skills can be taught within a curriculum context rather than a standalone ‘study skills workshop’ in order for students to find them
relevant. The spiral Mathematics curriculum thus lends itself well to the use of self-assessment. When students regularly self-assess, they gain insights into their learning which provides regular and timely descriptive feedback to guide their learning. Activities that promote self-assessment can stimulate discussions between learners and with such repeated opportunities to reflect on their learning, students’ future learning may then become more goal-directed and purposeful as they construct theories about the tasks, strategies, and meaning of schoolwork that are sensible and personal (van Kraayenoord & Paris, 1997). Table 1 summarises key questions that can be used to engage learners in self-regulatory behaviour in the primary mathematics classrooms. The phases and processes of self-regulation are adapted from Zimmerman (2000).

In the next three sections, we give examples of how SRL can be cultivated in daily mathematics lessons and mathematics units.

3 SRL for Learning Number Facts

3.1 Pre-action: Set goals.

At the beginning of each mathematics lesson, learning objectives, learning intentions, learning targets or success criteria can be co-constructed with the students for students to gain greater ownership and responsibility of their learning. An example of a learning target would be: To multiply and divide within the multiplication tables of 4. The success criteria would be I can commit to memory the 4 times table.

Although rehearsal is an effective procedure for tasks that require rote memorization, “rehearsal that rotely repeats information does not link information with what one already knows. Nor does rehearsal organize information in hierarchical or other fashion” (Schunk, 2004, p. 228). Retrieval of the facts may thus be difficult after some time. The multiplication facts can be grouped and organized into meaningful schemes before using rehearsal to memorize the facts. Pictures or mental images can be used to help students make sense of the multiplication facts to aid retrieval of the facts after some time (see Figure 2 for example of the 4 times table).
Developing Self-Regulated Learners

Table 1
Key questions to engage learners in self-regulatory behavior in the primary mathematics classrooms

<table>
<thead>
<tr>
<th>Phases and processes of self-regulation</th>
<th>Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Pre-action phase: Set goals</strong></td>
<td>(1) What are your goals for the learning situation?</td>
</tr>
<tr>
<td>(1) Goal setting</td>
<td>(2) What strategies/methods are appropriate for the learning situation?</td>
</tr>
<tr>
<td>(2) Strategic planning</td>
<td>(3) What are your motivations to learn?</td>
</tr>
<tr>
<td>(3) Self-motivational beliefs</td>
<td></td>
</tr>
</tbody>
</table>

**Action phase: Self-control, monitor and regulate**

| (1) Self-control                        | (i) How can you stay focus? |
| (i) Attention focusing                  | (ii) Describe how will you proceed to execute a task? |
| (ii) Self-instruction                   | (iii) What strategies can you use? |
| (iii) Task strategies                   | Study / learning strategies [e.g. note taking, test preparation summary etc.] |

| (2) Self-observation                    | Did you track your learning and performance? |
| (i) Self-recording                      | Did you record your learning? |
| (ii) Self-experimentation               | Did you track if the strategy works, why it works or did not work? |

| Post-action phase: Self-reflection      | (i) What have you achieved so far [against your goal]? |
| (1) Self-judgment                      | What are you still unclear about? |
| (i) Self-evaluation                    | (ii) What strategy helps you to learn / perform? |
| (ii) Causal attribution                 | What is the biggest help along the way in achieving your goal? |
|                                         | What is the biggest challenge that you face in achieving your goal? |

| (2) Self-reaction                       | (i) Are you satisfied with your performance? |
| (i) Self-satisfaction                   | What are you satisfied with? |
| (ii) Adaptive/ defensive inferences     | (ii) How can you do better? |

Skip counting together with the fingering method is one strategy that young children can use to commit the times table into memory. As shown in Figure 2, both hands are placed in front of the learner with the palms facing down. Hold down all the fingers. Each of the ten fingers represents
a specific number of groups e.g. left hand: baby finger represents 1 group, ring finger represents 2 groups, middle finger represents 3 groups, index finger represents 4 groups and thumb represents 5 groups. For the 4 times table, to solve $2 \times 4$, hold up the baby and ring fingers. Then skip count in 4s, 4, 8. To solve $3 \times 4$ hold up the baby, ring and middle fingers. Then skip count in 4s, 4, 8 and 12.

Because the processes in the pre-action phase are dependent on students’ self-motivation beliefs (e.g. self-efficacy, outcome expectations, intrinsic value, and goal orientation), these processes may not be effective if students cannot motivate themselves to use them. “Many ways to enhance student motivation relate directly to perceptions of task value, including showing students how tasks are important in their lives…” (Schunk, 2004, p. 385). As such, incorporating methods to enhance students’ perceived value of ‘learning the times table’ will more likely motivate students to exercise self-regulation over their learning activities. The times table can be linked to real-world phenomenon where we use the multiplication facts constantly in everyday life e.g. different numbers of legs animals have, different number of wheels etc.

![Figure 2. Multiplication table of 4.](image)
3.2 *Action phase: Self-control, monitor and regulate*

To help students *focus their attention* on remembering the times table, “slow motion task execution to assist coordination” (Mach (1988) cited in Zimmerman, 2000, p. 19) between skip counting and fingering method and the times table. Students can verbalize the times table to improve their learning of the times table as they *self-instruct* the skip counting and fingering method. Students can also be encouraged to track their own performance by monitoring their fingers and skip counting. Immediate self-feedback can be obtained by using the pictures in Figure 2.

3.3 *Post-action phase: Self-reflection*

In this phase, students “*evaluate* their learning results and draw conclusions concerning further learning behaviour” (Leidinger & Perels, 2012, p.2). Students can ask themselves “Have I committed the 4 times table to memory?” Self-criteria can be used by students to compare their performance at various parts of the mathematics lesson. Students can be prompted to reflect whether (i) the skip counting and fingering method are helpful learning strategies (ii) they are satisfied with their learning of the 4 times table (iii) there is any need to modify the skip counting and fingering method to cater to their (individual) learning needs or whether a more effective strategy can be chosen and (iv) there is a need to shift their learning goals and revisit pre-requisite skills of recapping addition within 40 with regrouping for $16 + 4 = 20$ and $28 + 4$.

4 SRL for Computational Skills

4.1 *Pre-action: Set goals*

When learning to add three whole numbers for young children, they can be provided the opportunities to add using different strategies such as counting on and adding by making 10. The goal for a mathematics lesson here could be *I can add three 1-digit numbers using make ten strategy*. “Many people who are highly skilled at mental math have developed ways of decomposing larger numbers and composing them in various ways to
make their mental math easier. Several of these methods involve composing and decomposing tens” (Schwartz, 2008, p. 53).

For example, there are many ways to perform a quick mental calculation for $47 + 29 + 32$. One way is to first add 4 tens, 2 tens and 3 tens which gives 9 tens. Next, add 9 ones and 2 ones which gives 11 ones. Then add 11 ones and 7 ones which gives 18 ones. In total, we have 9 tens and 18 ones giving us 118 as shown by Schwartz. This strategy depends on number sense. Perhaps, one motivation for learning the make ten strategy could be to develop a good number sense to empower the students “to find ways to enable quick mental calculation” (Schwartz, 2008, p. 53).

4.2 Action phase: Self-control, monitor and regulate

Before students can exact the appropriate strategies for attaining their goals, they must first experience the different strategies that exist to aid their learning. These strategies also had to be experienced in a structured manner and related to subject content (Smith, 1998). The concrete-pictorial-abstract (C-P-A) approach can be used as a learning experience for the make ten strategy. Figure 3 shows an example of how the various representations can be connected in a mathematics lesson. In the make ten strategy, students add three 1-digit numbers in 2 steps by using number bonds and making 10 strategy. Self-control in the form of “attention focusing” (Zimmerman, 2000, p. 19) can be achieved by having students clearly indicating the decomposition and composition of numbers (See Figure 3, Abstract). This will help students to concentrate on the final numbers to add. Students can self-instruct by describing the steps in adding any three 1-digit number.
Students can make self-observations of their learning and performance by monitoring the number that was decomposed, the numbers that make ten after the decomposition and the numbers that will be finally added – “tracking themselves selectively at a detailed process level” (Zimmerman, 2000, p. 19). Immediate self-feedback can be obtained by using the concrete manipulatives in Figure 3. Thereafter, students can engage in self-experimentation to see if adaptations or changes in their strategy improve their learning and performance. For example, Figure 3 (Abstract) requires the students to compose numbers that are not directly next to each other. This may be a challenge to some of the students. Students can be encouraged in this phase to engage in self-experimentation (see Figure 4).
to test out different numbers to decompose so as to enhance their performance.

<table>
<thead>
<tr>
<th>8 + 6 + 5 = 8 + 10 + 1</th>
<th>8 + 6 + 5 = 10 + 4 + 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 1</td>
<td>2 4</td>
</tr>
</tbody>
</table>

\[= 19 \quad = 19\]

*Figure 4. Make ten strategy.*

4.3 Post-action phase: Self-reflection

In this phase, students can be encouraged to self-evaluate their learning against the goal for the lesson. They can be prompted to reflect on the following: What are you still unclear about in the make ten strategy? What is the biggest challenge in learning and using the make-ten strategy? What helps you to remember and use the make-ten strategy? The teacher can also have students compare and contrast the counting on strategy and make-ten strategy to assist students to engage flexibly and adaptively in strategy selection and developing their metacognitive knowledge about task-specific strategies (Butler, 2002).

5 SRL for each unit of instruction

5.1 Pre-action: Set goals

A grade record sheet for students to record their desired and expected scores before embarking on mathematical tasks assigned to them can be used to help students identify the *learning goals* they had to achieve in each unit of instruction. Actual scores achieved for each of the tasks can be recorded by the students for the students to track their progress. Figure 5 shows a revised grade record sheet for the worksheets in Teong’s (2011) teaching package for the unit on four operations (whole numbers) to introduce the concept of goal setting.
The worksheets in the teaching package were designed such that a specific strategy can be taught in each worksheet for students to solve the mathematics problems for that unit. The revised grade record sheet can be given to the students at the beginning of the mathematics unit. Simple descriptors can be provided to students to assist students to state more realistic expected scores (see Figure 6).

<table>
<thead>
<tr>
<th>Worksheet / Quiz Focus</th>
<th>Score</th>
<th>Margin Symbols</th>
<th>Comments (Include what’s next?)</th>
<th>Met</th>
<th>Not yet met</th>
<th>Please notice… What do you hope your teacher will notice about your learning progress in this topic?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Learning Objectives: Solving word problems involving 4 operations.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WS 1: Opportunity cost</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WS 2: Redistribution</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>WS 3: Double ‘if’ conditions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>WS 4: Insufficiency</td>
<td>Extra - shortage</td>
<td>Shortage</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

**My self-review**

What did I learn by working towards my goal?
What was my biggest obstacle?
What was the biggest help along the way?
I could have done better if….
What am I still unclear about for this topic?
How do I feel after completing this topic?

*Figure 5. Revised sample of grade record sheet from Teong (2011).*
5.2 Action phase: Self-control, monitor and regulate

The specific strategy to be learnt in each of the worksheets can be very helpful to help students focus in their problem solving efforts. As the teacher administers each worksheet, specific strategy was explicitly taught so that students can self-instruct as they solve the mathematics problems. The worksheets contains a series of strategies to be learnt and can be organized meaningfully (e.g. simple to complex problems).

Students can be encouraged to indicate and record whether the worksheet is challenging or manageable by the student using the grade record sheet. Symbols such stars and ticks are some observable symbols used by some mathematics teachers. The margin symbols column in Figure 5 provides students and their teachers the opportunities to track their progress, e.g. keep track of the tasks that are challenging to them, and monitor their performance and progress for such tasks. Students can also be encouraged to engage in self-experimentation and test out whether alternative strategies will improve their performance.
5.3 Post-action phase: Self-reflection

The table in Figure 5 is an example of how students can go through the continual process of self-evaluation if the various strategies to solve problems involving 4 operations were achieved. The table in Figure 5 and criteria in Figure 6 can also be used by students to reflect why their expected scores did not match their actual scores. Students can next record their suggestions on the ‘comments’ column on how they could improve their grade in future tasks.

As the students progressed through each worksheet, they can use the table in Figure 5 to assess their own learning progress and tick the appropriate boxes to assess if they had acquired the strategies to be learnt from each worksheet in the ‘met / not yet met’ column. The table can also help students see the links between the different strategies acquired from each worksheet. Students can also be encouraged to note down salient points about their learning progress that they would like their teacher to notice in the ‘Please notice’ column. In doing so, students will be able to do timely and consistent reflection of each piece of work rather than wait till the end of each topic to reflect.

Timely feedback given by the teacher for each piece of work helps students to reflect on their strategy implementation and outcome based on the teachers’ feedback. For example, the teacher can use samples of students’ work to share exemplars of various positive problem solving processes students use in their worksheets. Through such feedback, students will be able to see how valid their self-evaluations are and will be able to become more accurate in self-monitoring, evaluation and regulating as each topic progresses.

Proof prompts were included at the end of the record sheet for students’ self-reflection at the end of the unit (see Figure 5). Through these prompts, students can assess and reflect on their own learning based on evidence from individual pieces of worksheets completed. Upon reflection on what worked and what did not, students will be able to select more appropriate strategies to improve. The prompts in Figure 5 can also be specific to a mathematics topic to help the students identify the challenges they face. Students’ responses to the prompts can be very useful information for the teachers to understand what the students are struggling
with and plan for instructional materials to address those difficulties. Figure 7 is an example of a self-review for students to reflect on the challenges they face and the biggest help they receive at the end of the unit.

Figure 7. Self-review from Teong (2011).

6 Conclusion

The 21st century learner is one who is self-directed in his/her own learning and perseveres towards his goals. There must be a shift of focus from didactic downloading of mathematics concepts to empowering students in becoming confident and responsible learners. In this chapter, we gave examples of how the phases and processes of self-regulation can be used in the primary mathematics classroom to develop self-regulated learners. We believe that the development of effective forms of self-regulation will develop in our young one’s an adaptive capability to respond and function well in this world.

Teachers, however, must provide a structure and even model how to self-regulate (Zimmerman, Bonner & Kovach, 1996). Through role-
modelling self-regulating behaviours and setting up a structure for developing self-regulating behaviours, appropriate help could be rendered to students in developing self-regulated learning skills so that students are able to set realistic and achievable goals, plan task-related strategies and determine the criteria for success. This is so that students are very clear about “Where are you going?”, “How are you going?” and “Where to next?” (Hattie, 2008). Students can learn, from a young age, to control, monitor and evaluate their own progress as they attempt to grasp mathematical concepts. This will help develop in them transferrable self-regulated learning skills that will carry them through their years of continuous learning.

Acknowledgement

Figures 5, 6 and 7 are taken from Teong’s Master of Education dissertation at the National Institute of Education, Singapore, under the direction of Cheng Lu Pien. We are grateful to the teachers who shared their teaching ideas and resources with us. This chapter draws upon some materials from the keynote lecture delivered by the first author for the Mathematics Teachers Conference 2016.

References


Smith, C. M. (1998). Underprepared college students’ approaches to learning mathematics while enrolled in a strategy-embedded developmental mathematics course and while subsequently enrolled in a college-level mathematics course that did not purposefully emphasize the use of mathematics-specific learning strategies. (Unpublished doctoral dissertation). The Ohio State University, Columbus.


Chapter 18

Empowering Students’ Learning through Mathematical Modelling

Chun Ming Eric CHAN     Rashidah VAPUMARICAN
Kaiwen Vanessa OH     Huanjia Tracy LIU     Yew Hwee SEAH

Research has shown that if learning is to be seen as an empowering activity, then learning ought to be a discursive activity that involves social interactions and sense-making through connecting, adding and correcting between pieces of knowledge. Drawing from a group case study, this chapter discusses a modelling approach to empowering Primary 5 students in learning mathematics through engaging a Model-Eliciting Activity (MEA). The modelling approach shifts the notion of problem-solving to the creation of models seen as systems of relationships used to interpret a given real-world problem situation. Samples of students’ discourse excerpts and mathematical solutions are exemplified to infer their mathematical reasoning, exploring, metacognating, decision-making and interpreting behaviours, actions that we deem as empowered learning.

1 Introduction

Education ministries around the world are transforming their curriculums to take into consideration means to develop 21st century life skills. These include, for example, the abilities to innovate, adapt, communicate and synthesise information (Stohlmann, 2013). In Singapore, our local mathematics curriculum seeks to enhance student learning experiences
such that the “process of learning becomes more important than just what is to be taught and remembered” (MOE, 2013, p.8), and in so doing promotes skills and competencies that will make a better 21st century learner. One of the means to accomplish the said goal is to provide students with opportunities to work on mathematical modelling activities where students will learn to “deal with ambiguity, make connections, select and apply appropriate mathematics concepts and skills, identify assumptions and reflect on the solutions to real-world problems, and make informed decisions based on given or collected data” (MOE, 2013, p18). Such an approach implies a shift from the traditional form of problem-solving where students are used to getting one correct answer through direct arithmetic translations to one where students generate constructs by making sense of the problem situation through mathematising and generating a model or a mathematical solution that best represents the problem situation (Lesh & Doerr, 2000). In this regard, modelling is seen to advance existing practices of problem-solving (English & Sriraman, 2010) and empower students to explore mathematics through different contexts of mathematics use (Galbraith, 1995).

This chapter discusses the role of mathematical modelling in empowering students’ mathematics learning through students’ engagement in a model-eliciting activity (MEA). Samples of Primary 5 students’ discourse and mathematical solutions from a case study in mathematical modelling are used to exemplify five areas in terms of empowering students, namely, reasoning, exploring, metacognating, decision-making, and interpreting. This chapter concludes with some considerations for teachers who wish to conduct MEAs purposefully and meaningfully with their students.

2 Mathematical Modelling and Model-Eliciting Activities

There are various definitions and perspectives of mathematical modelling stemming from different theoretical frameworks and research agendas (Kaiser & Grunewald, 2015; Toh, 2010). Despite the differences, a common feature in mathematical modelling is that the learner is confronted with a real-world situation and has to formulate a model as a
Empowering Students’ Learning through Mathematical Modelling

A mathematical solution to represent and interpret the real-world situation (see Figure 1, (MOE, 2013, p.18)).

Figure 1. A generic mathematical modelling process

The process of mathematical modelling tends to be cyclical as learners will have to express, test and revise their models towards making their models a better solution of the real-world problem. The learning outcome is the generation of mathematical models which are mathematical representations or idealisations of a real-world situation. The process where the data, concepts, relations, conditions and assumptions are translated into a mathematical representation, known as mathematising, is
Empowering Mathematics Learners

central to mathematical modelling (Blum & Niss, 1991) for this is where mathematical relationships are established between variables towards depicting and interpreting the problem situation.

In this chapter, we adopt the modelling perspective that involves the use of a model-eliciting activity (MEA). MEAs are problem situations which are designed as simulations of real-life problem-solving situations to elicit powerful, sharable, and re-usable models which are descriptive, explanatory or mathematical solutions (or systems) as representations to real-world problems (Doerr & English, 2003). Unlike some literature that strictly distinguishes modelling from problem solving, the use of MEAs from a modelling perspective tends to be more adaptable and sees the modelling perspective to problem-solving as engaging students with non-routine problem situations that elicit the development of significant mathematical constructs and then extending, exploring and refining those constructs towards a generalizable system (or model) (Chan, 2009; Mousoulides, Sriraman, & Christou, 2007). Research involving children engaged in MEAs has revealed positive outcomes where the young learners were seen to create abstract mathematical representations or models ranging from data organization to ranking information based on weightings before being formally taught (English, 2006; English & Watters, 2005). Local case studies of primary school students have also revealed their abilities to generate models with MEAs (Chan, 2010; Chan, Ng, Widjaja & Seto, 2012).

3 Empowering Nature of Model-Eliciting Activities

Empowering learners is about letting the students take ownership of their own learning. We see that as an inherent pedagogic approach in reformed pedagogies where students become the focal point of the process where they have opportunities to influence activities, materials and pace of learning (Aliusta & Ozer, 2014; Lim, 2014). The teacher has to devolve a fair amount of authority to the students where what used to be lengthy periods of teacher-talk are replaced by highly engaged learning experiences. In this regard, the teacher plays the role of an active listener cum facilitator and allows students to make and express judgements where
their responses are valued. This in turn would enable the learners to have a sense of ownership of their success that resulted from their own abilities and applications (Mholo & Schafer, 2012).

In the mathematics classroom, what is seen as empowering learning may be summarised in the selected points taken from Anthony and Walshaw’s (2009) Educational Practice Series on Effective Pedagogy in Mathematics as follows: (i) providing students with opportunities to work both independently and collaboratively in making sense of ideas, (ii) providing learning experiences that enable students to build on existing proficiencies, interests, and experiences, (iii) providing worthwhile mathematical tasks for students to view, develop, use and make sense of mathematics, (iv) providing support to students in creating connections between different ways of solving problems, between making representations and topics, and between mathematics and everyday experiences, (v) using a range of assessment practices to make students’ thinking visible and to support students’ learning, (vi) facilitating dialogue that is focused on mathematical communication, and (vii) selecting tools and representations to provide support and thinking. We concur with the positive outcomes based on the above points as enacted through effective pedagogy when students engage in MEAs.

From a modelling perspective, “thinking mathematically” is not just about mere computations but it involves thinking through what kind of situations they can describe mathematically and what models they have developed. Students will go through a process of selecting, filtering, organising and transforming information, and in so doing, express their conceptual systems as models (spoken language, written symbols, diagrams, metaphors, or computer-based simulations). The product is not just a solution, but descriptions, explanations, and justifications that reveal important aspects of the students’ thinking (Lesh & Doerr, 2000). The social context offers opportunities for constructive and critical engagement of mathematical ideas. Such a context also moulds the learners’ perceptions, understanding and mathematical realities and the knowledge that the learners develop can be seen as a product of, and anchored by the tools that drive the activity (Chinnappan, 2008). Moreover, the model-development process will generate productive mathematical discussions and simultaneously building on one another’s
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ideas when learners work cooperatively. In this sense, learning is equated with model development (Doerr & Arleback, 2015). With more space to think deeper about their model development, students exercise greater metacognition where there is a higher degree of reflection, critical thinking and making informed decisions. Research based on metacognitive processes that emerged from students’ engagement in MEAs reported that Grade 6 students showed greater awareness and Grades 7 and 8 students employed greater regulation and evaluation processes (Shahbari, Daher & Rasslan, 2014).

4 Illustrations of Empowered Student Learning with MEAs

The following sub-sections discuss the role of MEAs in empowering students in five main areas, namely, reasoning, exploring, metacognating, decision-making, and interpreting. The illustrations are drawn from a current research with a class of Primary 5 students involved in a MEA. The MEA (see Appendix) was adapted and modified from a model-eliciting task entitled Friendly Games used in an Australian school research for selecting men swimmers for the Commonwealth Games (English, 2010). The task was modified to suit the students’ age level and experience to ensure greater familiarity based on the local context of selecting their school swimmers for a National School Swimming Competition. A group of 4 students was video-recorded for the purpose of a case-study research and the other grouped students (non-target groups) were also involved in working on the same MEA but were not video-recorded.

4.1 Reasoning

Students had the opportunity to reason more explicitly as they discussed with their group members the models used to select the best swimmers. Unlike the traditional classroom practice where students were dominantly involved in independent seatwork and coupled with teacher-exposition as the main form of instruction, students engaged in MEAs were found to express their thoughts by thinking aloud, and the dynamics of the
discourse leads to greater instances of clarification, explanation and argumentation of the task at hand. The mathematics embedded in the task surfaced as a result of the collaborative discourse. It needs to be noted as well that not all students who work in groups would manifest a form of discussion where there is clear articulation of reasoning (Johnson & Johnson, 1990) and as such, the teacher has a part to play in helping and supporting the students towards eliciting clearer mathematical responses (Fraivillig, Murphy & Fuson, 1999). The following episode (from a fine-grained qualitative analysis) illustrates some of the mathematical reasoning behaviours such as analysing and comparing data, and drawing logical conclusions exhibited by students during the modelling process.

<table>
<thead>
<tr>
<th>Line</th>
<th>Person</th>
<th>Protocol</th>
<th>Teacher scaffolding and students’ mathematical reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tr</td>
<td>This is very fast right? (Referring to the “1s” in Adeline’s column). You just allocate the points? Tell me how you allocate the points?</td>
<td>Elicit</td>
</tr>
<tr>
<td>2</td>
<td>S1</td>
<td>We compare this (finger sweeps across the first event) with everyone, and then this (Adeline) is the fastest, so we give 1 point.</td>
<td>Analyses and compares data</td>
</tr>
<tr>
<td>3</td>
<td>Tr</td>
<td>So, you don’t even consider who is the second fastest in the inter-class swimming meet? You just want to see who is the fastest, that’s all? But you have to choose two in the end right? You need to recommend two.</td>
<td>Elicit</td>
</tr>
<tr>
<td>4</td>
<td>S1</td>
<td>So this one right, Nurul is the fastest, Harlindah is the slowest.</td>
<td>Analyses and compares data</td>
</tr>
<tr>
<td>5</td>
<td>Tr</td>
<td>So using this method, Adeline is 1, 1, 1. Nurul also has 1, 1, 1.</td>
<td>Support</td>
</tr>
<tr>
<td>6</td>
<td>S2</td>
<td>But she (Nurul) has one slowest (pointing at the M; most time).</td>
<td>Analyses and compares data</td>
</tr>
<tr>
<td>7</td>
<td>Tr</td>
<td>So do you choose Adeline and Nurul to be in the National Swimming Meet?</td>
<td>Elicit</td>
</tr>
<tr>
<td>8</td>
<td>S3</td>
<td>Because in the end, they both have 3 points but just that she (Nurul) has one slowest.</td>
<td>Draws logical conclusion</td>
</tr>
</tbody>
</table>

The excerpt highlights an initial instance when the students analysed and compared data of the swimmers’ timings (Lines 2, 4 and 6) towards determining their selecting of the choice of swimmers. The analysis and comparison was made on the basis of their establishing a model that
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involved a point system (as seen by the teacher’s protocols in Lines 1 and 5). Through the teacher’s questioning as a form of eliciting clearer responses, the students reasoning was shown to be more explicit than it could have been without the teacher intervention or the reasoning would not even have been captured. While the above excerpt has revealed that the students were engaged in analysing and comparing data, and drawing logical conclusion, other excerpts of coded protocols (not shown in this chapter) showed that students were also involved in justifying decisions, making conjectures and explaining mathematical concepts as part of the mathematical reasoning repertoire. By engaging in a MEA, students are empowered to articulate their mathematical thinking more.

4.2 Exploring

One relatable aspect of empowerment is to have the opportunity to explore solutions. MEAs provide affordances for students to explore different ways to represent the real-world situation. Given the real-world situation alongside a set of data, students may confront the task from various perspectives which will lead to generating a different model solution. In the said MEA, the students’ (case study and non-target groups) mathematization of the problem situation led them to generate the following systems (models), namely, (i) an elimination system as shown in Figure 2a, (ii) a point system as shown in Figure 2b, (iii) a graphical system as shown in Figure 2c and (iv) a system involving finding average timings as shown in Figure 2d.
Elimination System

Through analysing the data, some participants were “eliminated” as shown by the crosses against their names. Shortest timings were indicated with “I” and the longest timing with “M”.

*Figure 2a. An elimination system*

Point System

A matrix was used to capture shortlisted swimmers and a point system was devised. As shown, 4 points were awarded for the swimmer who was first in an event, 3 points for the second and so on.

*Figure 2b. A point system*
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Graphical System

Bar graphs are drawn to determine the accumulated timings of the swimmers before narrowing down to the four fastest swimmers. The average time was then taken.

Figure 2c. A graphical system

System Involving Averages

Students also used the concept of averages to determine the swimmers’ average timings to aid their choice of selection.

Figure 2d. A system involving averages
4.3 Metacognating

Compared to a structured word problem, the MEA is more complex and rich, and therefore is more cognitively demanding. According to Wilson and Clarke (2004), when tasks are more challenging, metacognition becomes essential in the process of solving the problem. They define metacognition as the awareness, evaluation and regulation of one’s thinking. We are convinced that engaging students in MEAs enables them to metacognate to a greater extent. Research based on elementary students’ endeavour in MEAs found an interplay of cognitive and metacognitive processes with students showing indications of awareness, evaluation and regulation behaviours (Shahbari, Daher & Rasslan, 2014). The episode below highlights an instance of the three aspects of metacognition (awareness, evaluation and regulation) displayed during the engagement of the current MEA on selecting swimmers. The teacher realized that the students had cancelled a chunk of mathematical solutions that they had penned and asked them about it.

Tr I see you all have cancelled this (pointing to the cancelled figures). You don’t like this idea? Why is that?

S2 Because it is not very accurate.

We inferred that the students had developed an awareness (what they realised) that their solution did not make good sense which led to the cancellation of their solutions. The students then made an evaluation (what they suspected was not working) with respect to why their solution “not very accurate”.

S2 Because if you look at this right (pointing to data in their working and the task sheet), their timings are all above 60. And then how can the average become 59?

The students had found that by averaging the timings of the swimmers, the average obtained (59 seconds) was less than the data values (above 60 seconds) which did not make sense. Through further questioning by the teacher about what could have gone wrong in the calculation, the students realized that they should not have used zero as a data item for swimmers
who did not participate in any event. This led the students to revise their thinking towards better reasoning in terms of applying the appropriate concept of averages (regulating) as seen below. The students divided the sum of the timings by 7 for swimmers who participated in 7 events instead of 8 when the swimmers were absent for one event.

S3 Should divide by the number of events she participated.

S4 We’ll take this plus this plus this (pointing to all the events) and divide 7. Ok. We’ll just try one.

The above episode is an example that points to the empowering nature of engaging in MEAs where students get the opportunity to express, test and revise their models, which from a metacognitive perspective, utilizes awareness, evaluation and regulation to assist in the process of learning as well as the attaining of solutions respective of the problem situation.

4.4 Decision-Making

In solving a real-world problem, it is hardly true that there would only be one solution to the problem. More often than not, there would be several different solutions as well as several means to come up with the solutions. People are then faced with decision-making actions towards making considered decisions. We are able to observe that in the case of students working on MEAs, the different models that the students generated result in situations where they could compare solutions and make informed decisions. They were be able to determine if one solution or model was a better fit than another as well as use the models to validate their choice. For example, in Figure 3, the group reported that “the (greater) the number of points the swimmers have, the lower the average timing is. Although Nurul Aini had 3 points, and 1 M (highest timing), she still accumulated the second most amount of points as she had 1 M. Looking back to the average, our 1st plan is correct but the second plan is more accurate as we used the actual timing. The 2nd plan was the reinforcement of the first plan.” Through comparing the outcomes of their point-system model with the outcome of the model using averages, the students were able to make
the considered decision of recommending Adeline Lim and Nurul Aini as the swimmers to represent the school.

![Image of students' explanation]

Figure 3. Students explain their decisions

4.5 Interpreting

By interpreting, we mean that students are given the opportunity to relate their findings back to the problem situation. The students had already generated their models, and they had to make sense of them by way of presenting the solution of the real-world context (MOE, 2012). This presents an opportunity for students to articulate their strategies and solutions as they make connections to the real world. As an example, the following excerpts depict how the group established the point system model and the average system model with reference to Figure 4.

“For this, we see who swam the fastest, then we gave 1 point. And then for the M, it is for the swimmers who swam the slowest. From this (chart), we say that Adeline swam the fastest 3 times, and Nurul Aini was the fastest 3 times but slowest 1 time. However, we thought that this (model) may not be so accurate because of this (points to the M), so we did the average ... our second plan to find...”
the average. So our second plan is actually linked to the first plan, so our first plan was correct.”

The excerpt expressed how the students allocated points to the swimmers and they being mindful that the system may not be so reliable, so they had to come up with a second model or “plan” as they called it. The students then elaborated how they worked on averages as their second plan.

“For the first time, we calculated everything (sum of data) and then divided by 8 although one of them did not participate. And then we found that the average did not make sense because all (the timings) is like above 60 and then suddenly their average becomes 59. So we changed it...the column that where the person did not participate, we will not count and divide by 7. For instance, Adeline, she had one DNP so we did not count and we divided by 7, we did not count the DNP.”

Figure 4. Students interpreting their solutions

We see that by providing students with opportunities to interpret the solutions, they learn to draw conclusions with respect to how the solutions address the original problem situation. Moreover, because the models are
to be shared with and used by others, they need to hold up under the scrutiny of other students who listen and have questions directed at them about the solutions (English, 2004).

5 Discussion

From what has been observed based on the students’ engagement in MEAs, we believe that MEAs are an excellent platform for students to work collaboratively in small groups to participate in the meaningful completion and solving of modelling problems. The aforementioned areas exemplified the richness of the students’ involvement where they had opportunities to reason mathematically by working on constructs and making connections between them as they generate conceptual systems or models. The sense-making process involves mathematising to transform what they understood from a real-world context into some conceptual systems involving mathematics that can be used to interpret the real-world situation. In so doing, the process becomes exploratory, where different models are generated for testing and for comparisons. According to English (2006, p.319), students “displayed many iterative cycles of expressing ideas, selecting and trailing factors, and creating, testing and revising models” and as exemplified earlier, a high degree of metacognition was exercised as depicted through their awareness, evaluative and regulatory behaviours when they compared and tested their models. Furthermore, the students had to make decisions about their models and interpret their models as a means of justifying how they arrive at their decisions. Throughout, the teacher’s role was to elicit, support and extend students’ thinking (Fraivillig, Murphy & Fuson, 1999) during the activity. Such a pedagogic platform serves to empower students in their learning of mathematics and solving of problems. They are in contrast to the traditional classroom instructional approach where the learning is said to be dominantly teacher-directed and where students have little opportunities to exercise reasoning through communication, exploration, deep metacognition, decision-making and interpretation. The students can be said to be engaged in an activity that prepares them for future-oriented learning where mathematical knowledge and skill skills related to
processing information, making conjectures, developing representational fluency, making evaluations and resolutions are valued.

While what has been reported may sound attractive in the light of what the students are capable of or could do when engaged in a MEA, it needs to be noted that a well-implemented MEA required conscientious forward planning between the researcher and the teacher-facilitators prior to conducting the MEA. The research experience we have acquired now allows us to share some implications that may be worth noting should teachers wish to carry out MEAs with their students. The implications are as follows.

a) Not all groups may be able to articulate their reasoning well unless the teacher plays the role of one who scaffolds the learning by asking questions to help students clarify their thinking as well as to extend their thinking (Wilson and Clarke, 2004). Revoicing is also a strategy that the teacher facilitator needs to use often for students to determine if their own reasoning that has been articulated is clearly understood by members, and this offers opportunity for the students to clarify their ideas and solutions.

b) Teachers need to keep in mind the notion of empowering students to take ownership of their own learning. It may be tempting for teachers to suggest and lead on when students appear to get stuck, and this may undermine the good intention of students generating their own solutions and deviate from the purpose of having MEAs.

c) It must be acknowledged that with a large class size, it would be better to make arrangements to have two or more teacher-facilitators to be able to sit with the groups of students and listen to their discussion intently for a while so that other groups that need some assistance would not feel neglected for too long a time if there is only one teacher-facilitator. The teacher-facilitators should make known to the students that their presence is to want to know more about what the students are doing and thinking and thus offer support, feedback and scaffolding rather than to formally assess if the students know or not know their mathematics.
d) It needs to be noted that the students who participated in the said MEA were all novice modellers. Some of the initial models generated tended to be naïve and unsophisticated. However, with appropriate teacher-scaffolding, the students revised and re-tested their models which resulted in an improved model or solution, and this is a motivational aspect in the sense that the students got to see themselves making progress.

e) Finally, if teachers are new to what MEAs are about, it is best to work with someone who has the experience and expert knowledge in the said domain as part of the professional development for teachers in embarking on MEAs. This collaborative experience has the advantage of working to design appropriate MEAs as well as develop a clear conception of what mathematics and models the students might be using or working on during the activity. This will indeed help the teachers greatly during their facilitation of the activity.

6 Concluding Remarks

In this chapter, we highlighted the potential of engaging students in MEAs as platform for empowering student learning. Examples from a classroom research are used to show what the students are capable of doing to a high degree with respect to reasoning, exploring, deciding, metacognating and interpreting, and with implications for enhancing student development in these areas further should teachers resolve to embark on MEAs. Although some of the findings reported in this chapter come from one case study research, and should not be generalised, many other such qualitative studies have been reported in literature on various developmental aspects of student learning with MEAs. There is certainly great potential for Singapore mathematics classroom practices to engage students in MEAs for a greater holistic development of student mathematics learning.
References


Empowering Students’ Learning through Mathematical Modelling


Selection of Swimmers for National Swimming Competition

Your school is one of the schools in Singapore that nurtures young swimmers for local swimming competitions. Currently, your school has identified 7 top swimmers for selection to represent your school in the National Swimming Competition. The best timings of the 7 swimmers in the 100m freestyle events are shown in the given table. Only 2 swimmers are to represent the school. Your group is part of the selection team to identify the two swimmers. Make use of the table given to you. Discuss how you go about developing a method to select the two most-suited swimmers. Write a report to make your recommendations by saying who are selected and why they are selected. Your reasons must include the methods you have developed.

<table>
<thead>
<tr>
<th>Competition</th>
<th>Name</th>
<th>Adeline Lim</th>
<th>Nurul Aini</th>
<th>Nancy Wen</th>
<th>Devi</th>
<th>Shanti</th>
<th>May Pereira</th>
<th>Harlin dah</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interclass Swimming Meet</td>
<td>65.48</td>
<td>67.52</td>
<td>68.63</td>
<td>66.81</td>
<td>72.45</td>
<td>67.54</td>
<td>DNP</td>
<td></td>
</tr>
<tr>
<td>D Division Championship</td>
<td>68.12</td>
<td>68.03</td>
<td>70.18</td>
<td>69.08</td>
<td>71.03</td>
<td>70.22</td>
<td>71.26</td>
<td></td>
</tr>
<tr>
<td>School Swim Race</td>
<td>66.65</td>
<td>64.88</td>
<td>DNP</td>
<td>64.95</td>
<td>68.11</td>
<td>65.82</td>
<td>68.13</td>
<td></td>
</tr>
<tr>
<td>School Cluster Swimming Meet</td>
<td>70.03</td>
<td>68.67</td>
<td>69.55</td>
<td>70.16</td>
<td>67.34</td>
<td>70.86</td>
<td>66.52</td>
<td></td>
</tr>
<tr>
<td>School Swimming Heats</td>
<td>DNP</td>
<td>65.93</td>
<td>68.02</td>
<td>71.23</td>
<td>67.10</td>
<td>68.91</td>
<td>69.08</td>
<td></td>
</tr>
<tr>
<td>Laser Lane Competition</td>
<td>64.88</td>
<td>71.06</td>
<td>DNP</td>
<td>65.81</td>
<td>DNP</td>
<td>65.77</td>
<td>66.07</td>
<td></td>
</tr>
<tr>
<td>South Zone Finals</td>
<td>66.90</td>
<td>67.11</td>
<td>69.98</td>
<td>69.94</td>
<td>70.62</td>
<td>69.98</td>
<td>67.74</td>
<td></td>
</tr>
<tr>
<td>Charity Swim Competition</td>
<td>71.77</td>
<td>71.57</td>
<td>73.47</td>
<td>68.22</td>
<td>66.47</td>
<td>68.12</td>
<td>70.61</td>
<td></td>
</tr>
</tbody>
</table>

DNP – Did Not Participate
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